The above formula is not quite correct because we have not included the decaying nature of the state $|a\rangle$. If $E_a \to E_a - i\Gamma_a/2$, then

$$\phi_a(\vec{r},t) = \phi_a(\vec{r})\, e^{-i(E_a t - i\Gamma_a t/2)/\hbar}$$

and $\phi_a^* \phi_a = e^{-\Gamma_a t/\hbar}$

so that probability density decays exponentially with lifetime $\hbar/\Gamma$.

~~Then~~ $\cancel{P_{a\to b} = \frac{2\pi}{\hbar^2} |\langle b|H|a\rangle|^2 \frac{1}{\pi} \mathrm{Re}\left(\frac{\Gamma_a/2}{(\omega_{ab} - i\Gamma_a/2)}\right)}$

$$\cancel{P_{a\to b} = \frac{2\pi}{\hbar^2} \mathrm{Re} \sum_{x,c} \frac{\langle b|H|a\rangle\langle a|H|x\rangle\langle x|H|c\rangle\langle c|H|b\rangle\, \omega_{ax}^2}{(E_b - E_a + i\Gamma_a/2)(E_x - E_a + i\Gamma_a/2)(E_x - E_b)(E_c - E_b)} \times \delta(\omega_{ax})}$$

$\cancel{= \frac{2\pi}{\hbar^2} \mathrm{Re}}$

$$P_{a\to b} = \frac{2\pi}{\hbar^2} |\langle b|H|a\rangle|^2 \frac{1}{\pi} \left| \frac{\Gamma_a/2}{(\omega_{ab} - i\Gamma_a/2)} \right|^2$$

Then

$$[illegible] = \frac{2\pi}{\hbar^2} |\langle b|H|a\rangle|^2 \frac{1}{\pi} \frac{(\Gamma/2)}{\omega_{ab}^2 + (\Gamma/2)^2}$$

Lorentzian line shape.

<u>Application to Radiation</u>

$$H = H_a + H_r + H_i$$

H_a = atomic hamiltonian
H_r = free field "
H_i = interaction.

In non-rel. approx.

$$H = \frac{1}{2m_N}\left(\underset{\sim}{p}_N - \frac{Ze}{c}\underset{\sim}{A}(\underset{\sim}{r}_N)\right)^2 + Ze\,\phi(\underset{\sim}{r}_N)$$

$$+ \frac{1}{2m}\sum_i\left[\underset{\sim}{p}_i + \frac{e}{c}\underset{\sim}{A}(\underset{\sim}{r}_i)\right)^2 - e\,\phi(\underset{\sim}{r}_i)\Big] + H_F$$

Note - electronic charge, = -e, not (+e) as ~~in Hamiltonian~~ before.

$$H_a = \frac{P_N^2}{2m_N} + \frac{1}{2m}\sum_i p_i^2 + Ze\,\phi(r_N) - e\sum_i \phi(r_i)$$

$$H_i = -\frac{Ze}{m_N c}\left\{\underset{\sim}{A}(\underset{\sim}{r}_N)\cdot \underset{\sim}{p}_N\right\}$$

$$+ \frac{e}{mc}\sum_i\left\{\underset{\sim}{A}(\underset{\sim}{r}_i)\cdot\underset{\sim}{p}_i\right\}$$

$$+ \frac{Z^2e^2}{2m_Nc^2}A^2(\underset{\sim}{r}_N) + \frac{e^2}{2mc^2}\sum_i \underset{\sim}{A}^2(\underset{\sim}{r}_i)$$

with $\nabla\cdot\underset{\sim}{A} = 0$.

In absence of interaction the free photon states satisfy

$$H_F\,|N_\lambda\rangle = U_{N_\lambda}\,|N_\lambda\rangle$$

λ = set of photon numbers specifying photon eigenfun.

Similarly $H_a|n\rangle = E_a|n\rangle$

$|n\rangle$ = free atomic eigenfun.

The states $|n\nu\rangle = |n\rangle|\nu\rangle$ are eigenfunctions of $H_F + H_a$ and represent the state of combined system in absence of interaction. The interaction H_i mixes the states $|n\nu\rangle$, leading to transitions, but the $|n\nu\rangle$ states form a complete set.

Then $P_{a\to b} = \frac{2\pi}{\hbar^2} |\langle a | H_H | b\rangle|^2 \delta(\omega_{ab})$

becomes

$$P_{n\nu\to n'\nu'} = \frac{2\pi}{\hbar^2} |\langle n\nu | H_i | n'\nu'\rangle|^2 \delta(\omega_{nn'} - \omega_{\nu\nu'})$$

H_i contains terms like $\vec{p}\cdot\vec{A}$. We first evaluate the matrix elements of $\vec{A}$ between the photon states $|\nu\rangle$ and $|\nu'\rangle$. In no. rep. $\vec{A}$ is an operator. In notation of M. Ajinashima

$$A_\mu = (\vec{A}, \frac{i\Phi}{c}) \qquad \text{4-vector}$$

$A_\mu A_\mu$ is invariant under Lorentz T's.

$x = (\vec{r}, itc)$, $k_\mu = (\vec{k}, i\frac{\omega}{c})$

Expand

$$A_\mu(\vec{x}) = \frac{1}{\sqrt{\Omega/4\pi}} \sum_{k,q} \frac{1}{\sqrt{2\omega}} e_\mu^{(q)} [c_{kq} e^{ikx} + c^\dagger_{kq} e^{-ikx}]$$

$$kx = \vec{k}\cdot\vec{r} - \omega t$$

c_{kq} = annihilation operator.
$c^\dagger_{kq}$ = creation operator

such that $[c_{\vec{k}q}, c^\dagger_{\vec{k}'q'}] = \delta_{\vec{k}\vec{k}'}\delta_{qq'}$

Put $\lambda = (\vec{k}, q)$

$$c_\lambda |N_\lambda\rangle = \sqrt{N_\lambda}\, |N_\lambda - 1\rangle$$

$$c^\dagger_\lambda |N_\lambda\rangle = \sqrt{N_\lambda + 1}\, |N_\lambda + 1\rangle$$

If one photon is absorbed from the field, then we are interested in the matrix element

$$\langle n' \, N_\lambda - 1 | H_i | n^* \, N_\lambda \rangle = \langle b | H_i | a \rangle$$

$$\langle N_\lambda - 1 | c_\lambda^* | N_\lambda \rangle = \sqrt{N_\lambda}$$

$$\therefore \langle n' N_\lambda - 1 | H_i | n^* \, N_\lambda \rangle = \frac{\sqrt{N_\lambda}}{\sqrt{\Omega}} \frac{1}{\sqrt{2\omega}} \langle n' | h_\lambda | n^* \rangle$$

where $h_\lambda = \frac{e}{m} \sum_i \hat{e}^{(q)} \cdot p_i \, e^{i \vec{k} \cdot \vec{r}_i}$

$$\simeq \frac{e}{m} \sum_i \hat{e}^{(q)} \cdot p_i \quad \text{for E1 transitions.}$$

$$= \frac{e}{i} \, \omega_{nn'} \sum_i \langle n' | \hat{e}^{(q)} \cdot \vec{r}_i | n^* \rangle$$

in length form.

$$\therefore P_{nN_\lambda \to n' N_\lambda - 1} = \frac{4\pi^2 N_\lambda \omega_{nn'} e^2}{\Omega \, \hbar} \left| \langle n' | \hat{e}^{(q)} \cdot \vec{r}_i | n \rangle \right|^2 \times \delta(\omega_{nn'} - \omega)$$

For spontaneous emission,

$$P_{nN_\lambda \to n' N_\lambda + 1} = \frac{4\pi^2 (N_\lambda + 1) \omega e^2}{\Omega \, \hbar} \quad \text{"}$$

To obtain the actual transition rate, the above must be multiplied by the final state density and integrated over ω. $A_{n \to n'} = \int P_{nN_\lambda \to n'N_{\lambda+1}}(\omega) \rho(\omega) \, d\omega$.

Assume that the volume Ω is a box of sides a, b, c.

If $\hat{e}^{(q)} e^{i\vec{k}\cdot\vec{r}}$ vanishes on boundaries,

$$k_x = \frac{2\pi n_x}{a}, \quad k_y = \frac{2\pi n_y}{b}, \quad k_z = \frac{2\pi n_z}{c}.$$

Divide Ω into cells with sides
$\frac{2\pi}{a}, \frac{2\pi}{b}, \frac{2\pi}{c}$.

Volume of cell $= (2\pi)^3/abc$.

No. of cells between k and $k+dk$
is $\frac{4\pi k^2 dk}{(2\pi)^3}\Omega = \frac{\Omega k^2 dk}{2\pi^2}$

$$= \frac{\Omega \omega^2 d\omega}{2\pi^2 c^3} \quad \times 2 \text{ polarizations}$$

$$\therefore \rho(\omega) = \frac{\Omega \omega^2 d\omega}{\pi^2 c^3}$$

and $A_{n\to n'} = \frac{4\omega^3 e^2}{c^3 \hbar} |\langle n' | \hat{e} \cdot \vec{r}_i | n \rangle|^2_{av}$

$$= \frac{4}{3} \frac{\omega^3 e^2}{c^3 \hbar} |\langle n' | \vec{r}_i | n \rangle|^2$$

go to p. 96

for discussion of oscillator strength.

Absorption $\qquad \omega = 2\pi\nu$

$u(\omega)d\omega = \frac{N_\omega}{\Omega} \times (\hbar\omega)d\omega$ = rad. density in freq. interval $d\omega$.

$u(\nu)d\nu = \left(\frac{2\pi N_\omega}{\Omega}\right) h\nu d\nu$ = " in $\nu \to \nu + d\nu$.

i.e. $\frac{N_\omega}{\Omega}(\hbar\omega) = \frac{du}{d\omega} = \frac{1}{2\pi}\frac{du}{d\nu}$

$\therefore P_{n N_\nu \to n' N_\nu - 1}$

$$= \int_0^\infty 2\pi u(\nu)\, \delta(\nu_{nn'} - \nu) \frac{e^2}{\hbar^2} |\langle n' | \hat{e}^{(g)} \cdot \vec{r}_i | n \rangle|^2$$

$$= u(\nu_{nn'}) \frac{2\pi e^2}{\hbar^2} |\langle n' | \hat{e}^{(g)} \cdot \vec{r} | n \rangle|^2$$

$$= u(\nu_{nn'}) B_{n \to n'} \, ,$$

Theory of Radiation

1. The harmonic oscillator

- Classical Hamiltonian is

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

$\omega = 2\pi\nu$, ν = frequency of oscillator.

ie $P.E. = \frac{1}{2} k x^2$

and $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$$\therefore \; PE = \frac{1}{2m}\left(m^2\omega^2 q^2\right)$$

Introduce variable $\eta = \frac{1}{\sqrt{2m\omega}}\left(p + im\omega q\right)$.

$[\eta^\dagger; \eta] = 1$ $\qquad \eta^\dagger = \;"\; \left(p^* - im\omega q^*\right)$

~~Then~~ Regard η as a dynamical variable.
The Heisenberg equ'ns. of motion are

$$\dot{q}_t = [q_t, H] = p_t/m$$

$$\dot{p}_t = [p_t, H] = -m\omega^2 q_t .$$

Then $\dot{\eta}_t = \frac{1}{\sqrt{2m\omega}}\left(\dot{p}_t + im\omega \dot{q}_t\right)$

$$= \frac{1}{\sqrt{2m\omega}}\left(-m\omega^2 q_t + i\omega p_t\right) = i\omega\eta_t$$

$$\therefore \; \eta_t = \eta_0 e^{i\omega t} .$$

By direct substitution, $\omega\eta\eta^\dagger = H - \frac{1}{2}\omega$

$$\omega\eta^\dagger\eta = H + \frac{1}{2}\omega$$

from $\eta^{+}H - H\eta^{T} = \omega\eta^{T}$,

Thus $\eta^{+}\eta - \eta\eta^{T} = 1$.

and $\eta^{T}\eta^{n} - \eta^{n}\eta^{T} = n\eta^{n-1}$.

By using arguments similar to those for finding eigenvalue spectrum of J^2, J_z, one obtains the eigenvalue spectrum of H.

e.g. let $|H'\rangle$ be an eigenket with eigenvalue H'.

Then $\omega\langle H'|\eta\eta^{+}|H'\rangle$

$$= \langle H'|H - \tfrac{1}{2}\omega|H'\rangle = (H' - \tfrac{1}{2}\omega)\langle H'|H'\rangle$$

Since $\langle H'|\eta\eta^{T}|H'\rangle \geq 0$ and $\langle H'|H'\rangle > 0$,

$$H' \geq \tfrac{1}{2}\omega.$$

$$H' = \tfrac{1}{2}\omega \text{ iff } \eta^{T}|H'\rangle = 0.$$

- What is $\eta^{T}|H'\rangle$?

$$H\eta^{T}|H'\rangle = \eta^{+}(H-\omega)|H'\rangle = (H'-\omega)\eta^{+}|H'\rangle$$

$\therefore$ $\eta^{+}|H'\rangle$ is also an eigenket with eigenvalue $H' - \omega$.

Similarly $H\eta|H'\rangle = (H'+\omega)\eta|H'\rangle$.

Let $|0\rangle$ be the lowest eigenket such that $\eta^{+}|0\rangle = 0 \implies H' = \tfrac{1}{2}\omega$.

Build up sequence of eigenkets

	H'
$\|0\rangle$	$\tfrac{1}{2}\omega$
$\eta\|0\rangle$	$\tfrac{3}{2}\omega$
$\eta^2\|0\rangle$	$\tfrac{5}{2}\omega$

~~$\eta^\dagger|0\rangle = 0$ is~~

~~$(p - im\omega q)|0\rangle = 0.$~~

These eigenkets are not necessarily normalized to unity. Assume $\langle 0|0\rangle = 1$

Then $\eta^\dagger \eta^n |0\rangle = n\eta^{n-1}|0\rangle + \eta^n \underbrace{\eta^\dagger |0\rangle}_{=0}$

$$\therefore \quad \eta^{\dagger n}\eta^n|0\rangle = n\,\eta^{\dagger n-1}\eta^{n-1}|0\rangle$$

$$\langle 0|\eta^{\dagger n}\eta^n|0\rangle = n\langle 0|\eta^{\dagger n-1}\eta^{n-1}|0\rangle$$

$$= n!\,\langle 0|0\rangle$$

The normalized eigenkets are $\frac{1}{\sqrt{n!}}\eta^n|0\rangle$.

An arbitrary ket $|x\rangle$ can be expanded

$$|x\rangle = \sum_{n=0}^{\infty} \frac{x_n}{\sqrt{n!}}\eta^n|0\rangle .$$

$|x_n|^2$ = probability of oscillator being in n'th. quantum state

The state $\eta^n|0\rangle$ contains n quanta. The above representation is called Fock's representation.

Schrodinger representation

$$\eta^\dagger|0\rangle = 0 \quad \text{is}$$

$$(p - im\omega q)|0\rangle = 0 .$$

In co-ord. rep.,

$$\langle q'|p - im\omega q|0\rangle = 0$$

or $\frac{\partial}{\partial q'} \langle q'|0\rangle + m\omega q' \langle q'|0\rangle = 0$

$\langle q'|0\rangle = \psi(q')$: wave function in co-ord. space.

The solution is $\langle q'|0\rangle = \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-m\omega q'^2/2}$

The n'th. eigenstate is $\eta^n|0\rangle$

$$\langle q'|\eta^n|0\rangle = \frac{1}{(2m\omega)^{n/2}} \langle q'|(p+im\omega q)^n|0\rangle$$

$$= \frac{1}{(2m\omega)^{n/2}} i^n \left(-\frac{\partial}{\partial q'} + m\omega q'\right)^n \langle q'|0\rangle$$

$$= \frac{1}{(2m\omega)^{n/2}} i^n \left(-\frac{\partial}{\partial q'} + m\omega q'\right)^n \left(\frac{m\omega}{\pi}\right)^{1/2} e^{-m\omega q'^2/2}$$

generates Hermite poly's.

$\frac{1}{(n!)^{1/2}} \eta^n|0\rangle$ is normalized eigenket.

<u>An Assembly of Non-interacting Bosons.</u>

- Suppose a single boson can exist in states $|\alpha^a\rangle, |\alpha^b\rangle, |\alpha^c\rangle, \cdots$.

A basis ket for describing u' bosons is

$$|\alpha_1^a \alpha_2^b \alpha_3^c \cdots \alpha_{u'}^q\rangle = |\alpha_1^a\rangle|\alpha_2^b\rangle \cdots |\alpha_{u'}^q\rangle$$

Since a boson state must be totally symmetric, apply symmetrizing operator

$$S = \frac{1}{(u'!)^{1/2}} \sum P$$

sum over $u'!$ permutations.

i.e. $\frac{1}{(n'!)^{1/2}} \sum P |\alpha_1^a \alpha_2^b \cdots \alpha_{n'}^q\rangle = S|\alpha^a \alpha^b \cdots \alpha^q\rangle$

Out of all possible states $|\alpha^{(1)}\rangle, |\alpha^{(2)}\rangle, \cdots$ the states $|\alpha^a\rangle, |\alpha^b\rangle, \cdots$ are occupied. If $a=b$, then the state $|\alpha^a\rangle|\alpha^b\rangle = |\alpha^a\rangle|\alpha^a\rangle$ is multiply occupied, with $n_a = 2$.

- Let $n_1', n_2', \cdots$ be the number of bosons in states $|\alpha^{(1)}\rangle, |\alpha^{(2)}\rangle$ etc.

If all the n_i''s are 0 or 1, then

$\langle \alpha^a \alpha^b \cdots \alpha^q | S^2 | \alpha^a \alpha^b \cdots \alpha^q\rangle = 1$

since each term in sum on left has only one non-vanishing contribution with terms on right and there are $n'!$ of them.

If ~~the~~ a state $|\alpha^{(i)}\rangle$ contains n_i' bosons, then there will be $n_i'!$ contributions on right for each term on left.

$\therefore \langle \alpha^a \alpha^b \cdots \alpha^q | S^2 | \alpha^a \alpha^b \cdots \alpha^q\rangle = n_1'! \, n_2'! \cdots$

We can regard the n_i''s as dynamical variables having eigenvalues $0, 1, \cdots n'$. Use the set of eigenvalues to label the states.

$(n_1'! \, n_2'! \cdots)^{-1/2} S|\alpha^a \alpha^b \cdots \alpha^q\rangle = |n_1' n_2' \cdots\rangle.$

If the total number of bosons is variable, then a complete set of boson states is

$$\begin{aligned}
&|\ \rangle &&= |0000\cdots\rangle \\
&|\alpha^a\rangle &&= |00\cdots 0 n_a 0\cdots\rangle && n_a = 1 \\
&S|\alpha^a\alpha^b\rangle &&= |0\cdots n_a \cdots n_b\cdots\rangle && n_a = 1, n_b = 1 \\
&S|\alpha^a\alpha^b\alpha^c\rangle && && \text{or } n_a = 2, n_b = 0. \\
&\quad\vdots
\end{aligned}$$

$a, b, c = 1, 2, 3, 4, \cdots$ in any combination.

If energy of state $|\alpha^a\rangle$ is $H(\alpha^a)$, then total energy is

$$\sum_a n_a H(\alpha^a)$$

Connection Between Bosons and Oscillators

- Consider a dynamical system consisting of several non-interacting harmonic oscillators.

- Each oscillator is described by an η and $\eta^\dagger$. Label oscillators by $1, 2, 3, \cdots$.

Then $\eta_a \eta_b - \eta_b \eta_a = 0$

$$\eta_a^\dagger \eta_b^\dagger - \eta_b^\dagger \eta_a^\dagger = 0$$

$$\eta_a^\dagger \eta_b - \eta_b \eta_a^\dagger = \delta_{ab}$$

Put $\eta_a \eta_a^\dagger = n_a$

$$\eta_a^\dagger \eta_a = n_a + 1$$

- We found before that

$$\omega \eta^\dagger \eta = H + \tfrac{1}{2}\omega$$

$$\omega \eta \eta^\dagger = H - \tfrac{1}{2}\omega$$

where H has eigenvalues $\frac{1}{2}\omega, \frac{3}{2}\omega, \cdots$

$\therefore\ n_\alpha = 0, 1, 2, \cdots$.

The n_α's form a set of commuting observables. Let $|0_\alpha\rangle$ denote ground state of α'th. oscillator. Then

$$|0_1\rangle |0_2\rangle |0_3\rangle \cdots \quad = |0\rangle$$

is the Fock representation for ground state of set of oscillators.

A state labelled by integers $n_1', n_2', n_3' \cdots$ is then

$$\eta_1^{n_1'} \eta_2^{n_2'} \eta_3^{n_3'} \cdots |0\rangle. = |n_1' n_2' n_3' \cdots\rangle$$

The above is a simultaneous eigenket of $n_1, n_2, n_3 \cdots$ having eigenvalues $n_1', n_2', n_3' \cdots$.

- The set of kets obtained by taking all possible values of $n_1', n_2', n_3' \cdots$ is orthogonal and complete.

Since $\langle 0| \eta^{\dagger n} \eta^n |0\rangle = n!$,

$$\langle n_1' n_2' n_3' \cdots | n_1' n_2' n_3' \cdots \rangle = n_1'!\, n_2'!\, n_3'! \cdots$$

The boson states and H.O. states are labelled by same quantum nos. and have the same normalization. We can thus set up a 1 to 1 correspondence

$$S\,|\alpha^a \alpha^b \alpha^c \cdots \alpha^q\rangle = \eta_a \eta_b \eta_c \cdots \eta_q |0\rangle$$

boson states $\qquad\qquad$ HO states.

.. ~~Each~~ There is one oscillator associated with each boson state (not boson)

no. of bosons in state "a"	H.O.	
0	$\|0\rangle$	ground state
1	$\eta_a \|0\rangle$	1'st excited state
2	$\eta_a^2 \|0\rangle$	2'nd " " .
⋮	⋮	

For a photon, the state a refers to definite momentum and polarization. (or angular momentum and parity).

- The correspondence means in particular that the boson kets $|\alpha^a\rangle$ and the corresponding η_a transform in the same way.

.. Suppose we transform to a different set of basis kets $|\beta^A\rangle$.

$$|\rangle, \quad |\beta^A\rangle, \quad S\,|\beta^A \beta^B\rangle \quad \cdots$$

$$|\beta^A\rangle = \sum_a |\alpha^a\rangle\langle\alpha^a|\beta^A\rangle$$

$|\beta^A\rangle = \eta_A |0\rangle$, where η_A is the corresponding oscillator operator in new rep. Then

$$\eta_A |0\rangle = \sum_a \eta_a |0\rangle\langle\alpha^a|\beta^A\rangle$$

$$\text{or} \quad \eta_A = \sum_a \eta_a \langle\alpha^a|\beta^A\rangle$$

Similarly, the $\eta_a^\dagger$ transform in the same way as the $\langle \alpha^a |$.

$$\langle \beta^A | = \sum_a \langle \beta^A | \alpha^a \rangle \langle \alpha^a |$$

and $\eta_A^\dagger = \sum_a \langle \beta^A | \alpha^a \rangle \eta_a^\dagger$.

An arbitrary ket $|x\rangle$ can be expanded

$$|x\rangle = \sum_a |\alpha^a\rangle \underbrace{\langle \alpha^a | x \rangle}_{\text{Fourier coef.}}$$ like Fourier analysis.

or, in terms of $|\beta^A\rangle$ set,

$$|x\rangle = \sum_A |\beta^A\rangle \langle \beta^A | x \rangle$$

where $\langle \beta^A | x \rangle = \sum_a \langle \beta^A | \alpha^a \rangle \langle \alpha^a | x \rangle$.

$\therefore$ the fourier coefs. $\langle \alpha^a | x \rangle$ transform in the same way as $\eta_a^\dagger$. The process of second quantization consists of replacing the coefs. $\langle \alpha^a | x \rangle$ by $\eta_a^\dagger$.

- The above provides the appropriate mathematical basis for describing an assembly of bosons. Boson statistics are contained implicitly in commutation relations.

Example $\quad U = 1$ boson operator.

$$\langle x | U | y \rangle = \sum_{a,b} \langle x | \alpha^a \rangle \langle \alpha^a | U | \alpha^b \rangle \langle \alpha^b | y \rangle$$

$$U_{op} = \sum_{a,b} \eta_a U_{ab} \eta_b^\dagger .$$ second quantized form of U.

Use to evaluate some specific matrix element $\langle \alpha^{(1)} | U | \alpha^{(2)} \rangle$

In second quantized form,

$$\langle \alpha^{(1)} | U | \alpha^{(2)} \rangle = \langle 0 | \eta_1^\dagger U_{op} \eta_2^* | 0 \rangle$$

$$= \langle 0 | \sum_{ab} \eta_1^\dagger \eta_a \, U_{ab} \, \eta_b^\dagger \eta_2 | 0 \rangle$$

$$\langle 0 | \eta_1^\dagger \eta_a = \langle 0 | (\underset{\overset{\shortparallel}{0}}{\eta_a \eta_1^\dagger} + \delta_{a1})$$

$$\eta_b^\dagger \eta_2 | 0 \rangle = (\underset{\overset{\shortparallel}{0}}{\eta_2 \eta_b^\dagger} + \delta_{b2}) | 0 \rangle$$

$$\therefore \langle \alpha^{(1)} | U | \alpha^{(2)} \rangle = \langle 0 | \sum_{ab} \delta_{1a} U_{ab} \delta_{2b} | 0 \rangle$$

$$= U_{12} \langle 0 | 0 \rangle = U_{12}.$$

The same technique is used if the initial and final states contain several bosons - commutation relations are used repeatedly to move η's to ~~my~~ left and $\eta^\dagger$'s to right.

- A laser corresponds to one or a few oscillators in highly excited states.