Examples.

1. Let G = group of integers under addition.

   H = sub-group of multiples of 4 under addition.

   Then $G = H + (1+H) + (2+H) + (3+H)$

2. $G = S_3$, $H = e, (12)$

   Then $G = H + (13)H + (23)H = H + H(13) + H(23)$

   $(13)H = (13), (13)(12)$
   $= (13), (123)$

   and $H(13) = (13), (12)(13)$
   $= (13), (132)$

   (in general $(1abc \cdots f) = (1f) \cdots (1c)(1b)(1a)$)

Conjugate Classes

- Elements a and b of G are said to be conjugate if there is another element u such that

$$uau^{-1} = b$$

Properties

1. Choose $u = e$. Then $eae^{-1} = a$
2. If $v = u^{-1}$, then $a = vbv^{-1}$
3. If $b = uau^{-1}$ and $c = vbv^{-1}$, then

$$c = vuau^{-1}v^{-1} = (vu)a(vu)^{-1} = waw^{-1}.$$

This establishes what is known as an equivalence relation.

Use $\equiv$ for "equivalent to". Then

1. $a \equiv a$
2. if $a \equiv b$ then $b \equiv a$.
3. if $a \equiv b$ and $b \equiv c$ then $a \equiv c$.

Elements conjugate to one another belong to the same conjugate class.

Examples.

1. abelian groups - every element is in a class by itself since $bab^{-1} = a$ for all a, b.

2. e is always in a class by itself. $e = aea^{-1}$

Class Properties

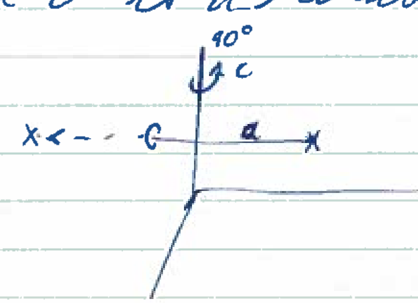
- Elements in the same class are, in some sense, of the same type. Specifically

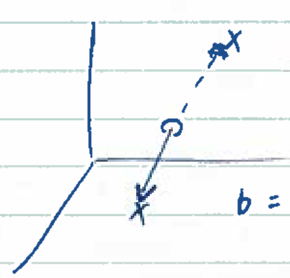
1. Elements in the same class have the same order. e.g. if $a^h = e$, ~~then~~ and

$$b = uau^{-1}, \text{ then } b^h = (uau^{-1})^h$$

$$= uau^{-1}uau^{-1} \cdots uau^{-1}$$
$$= ua^h u^{-1} = e.$$

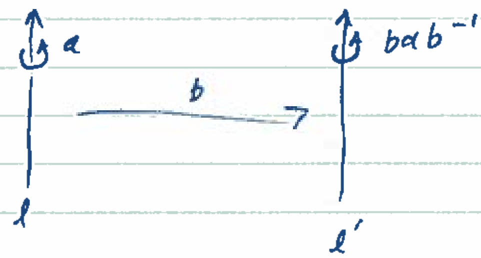
2. Spacial Transformations
- Suppose a is a reflection in a plane P and c is a rotation about some axis





$b = cac^{-1}$

or



3. Conjugate Permutations

Let $a = \begin{pmatrix} 1 & \cdots & n \\ a_1 & & a_n \end{pmatrix}$

$$b = \begin{pmatrix} 1 & \cdots & n \\ b_1 & & b_n \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_n \\ b_{a_1} & & b_{a_n} \end{pmatrix}$$

Then $bab^{-1} = \begin{pmatrix} a_1 & \cdots & a_n \\ b_{a_1} & & b_{a_n} \end{pmatrix}\begin{pmatrix} 1 & \cdots & n \\ a_1 & & a_n \end{pmatrix}\begin{pmatrix} b_1 & \cdots & b_n \\ 1 & & n \end{pmatrix}$

$$= \begin{pmatrix} b_1 & \cdots & b_n \\ b_{a_1} & & b_{a_n} \end{pmatrix}$$

ie make the substitutions $\begin{matrix} 1 \to b_1 \\ \vdots \\ n \to b_n \end{matrix}$ ie. $\begin{matrix} a_1 \to b_{a_1} \\ \vdots \\ a_n \to b_{a_n} \end{matrix}$

separately in the top and bottom rows of a.

eg. if $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix} = (12)(345)$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix} = (13524)$$

Then $bab^{-1} = \begin{pmatrix} 3 & 4 & 5 & 1 & 2 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix} = (34)(512)$

- Notice that the cycle structure is unchanged. i.e. elements with the same cycle structure belong to the same class.

e.g. S_3

e

$(12), (13), (23)$

$(123), (132)$

S_4

e

$(12), (13), (14), (23), (24), (34)$

$(12)(34), (13)(24), (14)(23)$

$(123), (132), (124), (142), (234), (243); (134), (143)$

$(1234), (1243), (1342), (1423), (1432)$

(1342)

- In general, resolve a permutation in S_n into cycles. Suppose there are

ν_1 - 1 cycles

ν_2 - 2 cycles

$\vdots$

ν_n - n cycles.

Then $\quad \nu_1 + 2\nu_2 + \cdots + n\nu_n = n. \qquad (1)$

Such a cycle structure is written

$(1^{\nu_1}, 2^{\nu_2}, \cdots n^{\nu_n}) \equiv (\nu)$ specifies a class.

Each integral solution to (1) determines a different class.

- The solutions can be arranged in a more illuminating form as follows.

Define
$$\begin{aligned} \nu_1 + \nu_2 + \nu_3 + \cdots + \nu_n &= \lambda_1 \\ \nu_2 + \nu_3 + \cdots + \nu_n &= \lambda_2 \\ \nu_3 + \cdots + \nu_n &= \lambda_3 \\ &\vdots \\ \nu_n &= \lambda_n \end{aligned}$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$n$$

$$\therefore \lambda_1 + \lambda_2 + \cdots + \lambda_n = n \,, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

Each partition of n into a sum of integers defines a class of S_n.

e.g. $5 = 2+2+1+0+0$ $\qquad (22100) = (221) = (2^2 1)$
$\qquad \lambda_1 + \lambda_2 + \lambda_3 + \cdots$

The inverse equations are
$$\begin{aligned} \nu_1 &= \lambda_1 - \lambda_2 &= 0 \\ \nu_2 &= \lambda_2 - \lambda_3 &= 1 \\ \nu_3 &= \lambda_3 - \lambda_4 &= 1 \\ & & 0 \end{aligned}$$

for the cycle structure $(\circ\circ)(\circ\circ\circ)$.

For the first few groups, the partitions are

S_1 $\quad (1)$
S_2 $\quad (2), (1^2)$
S_3 $\quad (3), (2,1), (1^3)$
S_4 $\quad (4), (3,1), (2^2), (21^2), (1^4)$

Note - the above division into classes applies only to the full group S_n. For example, the class (2^2) of S_4 contains

$$(12)(34)\,, \quad (13)(24)\,, \quad (14)(23)$$

because S_4 also contains the elements

$b = (23)$ such that

$$b\,(12)(34)\,b^{-1} = (13)(24)$$

and $b = (243)$ such that

$$b\,(12)(34)\,b^{-1} = (14)(23)$$

Recall that the four-group contains the elements e, $(12)(34)$, $(13)(24)$, $(14)(23)$, but now each element is in a class by itself since the group is abelian.

Number of Elements in a Class of S_n

- For a class with cycle structure $(\nu) = (\nu_1 \nu_2 \cdots \nu_n)$, the number of elements is

$$n_{(\nu)} = \frac{n!}{1^{\nu_1}\nu_1!\, 2^{\nu_2}\nu_2! \cdots n^{\nu_n}\nu_n!}$$

$n!$ = no. of ways of entering n numbers into
$(\cdot)(\cdot) \cdots (\cdot\cdot)(\cdot\cdot) \cdots (\cdot\cdot\cdot) \cdots$

$\nu_1!\,\nu_2! \cdots \nu_n!$ = no. of permutations of brackets among themselves
e.g. $(1)(2) = (2)(1)$

$1^{\nu_1} 2^{\nu_2} \cdots n^{\nu_n}$ = no. of equivalent rearrangements of numbers in brackets
eg. $(123) = (231) = (312)$.

3^{ν_3} ← no. of cycles of length 3.

↑ no. of equiv. re-arrangements

Invariant Subgroups

- A subgroup H is said to be "invariant" if it contains elements in complete classes.

i.e. for every element h_1 of H, and "a" of G there is another element h_2 of H such that

$$h_2 = a h_1 a^{-1}$$

This can be summarized symbolically by writing

$$a H a^{-1} = H. \qquad \text{all } a \text{ in } G.$$

i.e. $a H = H a$.

- In general, $a H a^{-1}$ is called the conjugate subgroup.

∴ a subgroup is invariant if its left and right cosets are equal. The set of elements H taken as a whole commute with any element a.

Simple group - has no invariant subgroups.
Semisimple - no invariant subgroup is abelian.

Factor Group

- Invariant subgroups can be used to form a new group called the factor group.

- Note that if H is invariant, then

$$(aH)(bH) = (aHb)H = ab(HH) = (ab)H.$$

i.e. the cosets associated with a and b have the same multiplication table as the elements ~~associated with~~ a and b themselves.

- $\mathcal{H}$ itself acts like an identity element since

$$\mathcal{H}(a\mathcal{H}) = a(\mathcal{H}\mathcal{H}) = a\mathcal{H}.$$

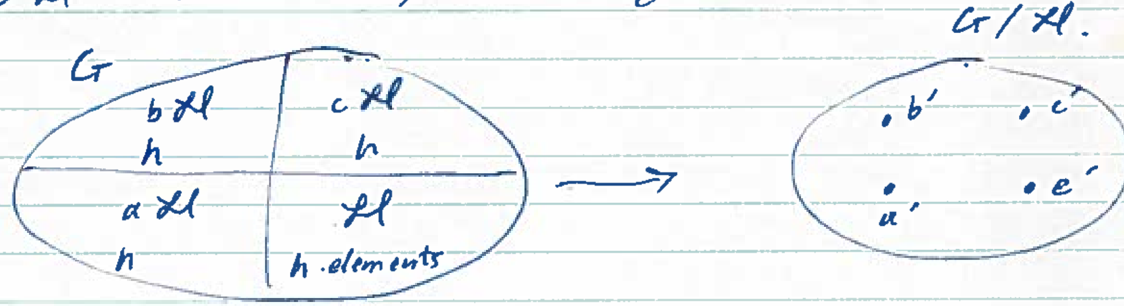
*

and $a^{-1}\mathcal{H}$ is the inverse of $a\mathcal{H}$.

∴ The cosets of $\mathcal{H}$ form a group under "coset multiplication" called the factor group $G/\mathcal{H}$.

Make the mapping

Elements of G		Elements of $G/\mathcal{H}$
invariant subgroup $\mathcal{H}$	$\longrightarrow$	e'
$a\mathcal{H}$	$\longrightarrow$	a'
$b\mathcal{H}$	$\longrightarrow$	b'
$c\mathcal{H}$	$\longrightarrow$	c'



- In each case, h elements of G map into one element of $G/\mathcal{H}$.
∴ mapping is a <u>Homomorphism</u>.

- In general, if $g = mh$ (m is the index of $\mathcal{H}$ in G), then $G/\mathcal{H}$ is of order $g/h = m$.

<u>Conversely</u> - in any homomorphism from one group G to another group G', the elements of G mapped into e' always form an invariant

subgroup H, and the other elements of G' correspond to the cosets of H in G. Note that if $E \to E'$ uniquely then the mapping is an isomorphism

Group Representations

Preliminary Remarks

- We will represent the group elements by matrices which operate in a linear vector space L.

i.e. if $\vec{x}$ and $\vec{y}$ are in L, then so are $\alpha \vec{x}$ and $\vec{x} + \vec{y} = \vec{y} + \vec{x}$.

- An n-dimensional vector space L_n is spanned by basis vectors $\vec{u}_1, \vec{u}_2, \cdots \vec{u}_n$ and

$$\vec{x} = \sum_i x_i \vec{u}_i .$$

Consider a linear mapping of L_n onto itself; i.e.

$$\vec{y} = \underset{\sim}{T} \vec{x} .$$

In the basis set $\vec{u}_i$, T is represented by a matrix T_{ij} such that

$$y_i = T_{ij} x_j$$

- Define a new basis set ~~by~~ $\vec{u}_i' = a_{ij} \vec{u}_j$

Then a fixed vector $\vec{x}$ is

$$\vec{x} = x_i \vec{u}_i = x_j' \vec{u}_j'$$

$$= x_j' a_{ji} \vec{u}_i$$

$$\therefore \quad x_i = x_j' a_{ji} = a^T_{ij} x_j'$$

Define $\vec{u} = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Then $\vec{u}' = \underset{\sim}{a} \vec{u}$

and $\vec{x} = \underset{\sim}{a}^T \vec{x}'$

or $\vec{x}' = (\underset{\sim}{a}^T)^{-1} \vec{x}$.

Thus, the equation $\vec{y} = \underset{\sim}{T} \vec{x}$ becomes in the new basis set

$$(\underset{\sim}{a}^T)^{-1} \vec{y} = \underbrace{(\underset{\sim}{a}^T)^{-1} \underset{\sim}{T} (a^T)}\,(a^T)^{-1} \vec{x}$$

$$\vec{y}' = \underset{\sim}{T}' \vec{x}'$$

$\underset{\sim}{T}$ and $\underset{\sim}{T}'$ are equivalent matrices. They specify the same vector mapping expressed in a different basis set!

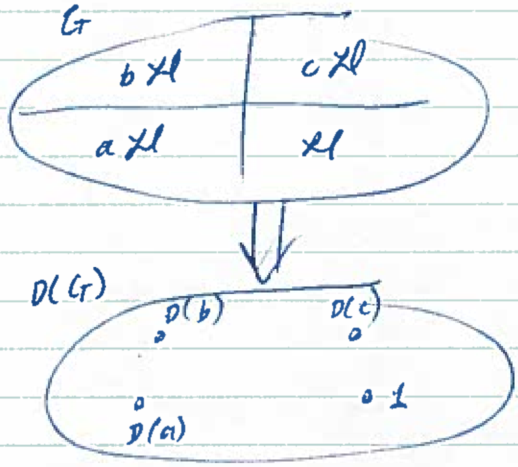
Put $\underset{\sim}{S} = (\underline{a}^T)^{-1}$. Then $\vec{y}' = \underset{\sim}{S} \vec{y}$

$$\underset{\sim}{T}' = \underset{\sim}{S} T \underset{\sim}{S}^{-1}.$$

Group Representations

- Suppose the set of vector operators $A, B, \cdots$ in a vector space L_n form a group.

It may be possible to establish a homomorphism between the elements of an arbitrary group G and the vector operators $A, B, \cdots$. In general



In general, if R and S are group elements ~~representations~~, then

$$D_{ij}(RS) = \sum_k D_{ik}(R)\, D_{kj}(S)$$

and $D_{ij}(E) = \delta_{ij}$

- Different representations are distinguished by a superscript μ

i.e. $D_{ij}^{(\mu)}(R)$ is the matrix representing R in the μ'th representation.

Equivalent Representations and Characters

- Recall that the matrix operators $\underset{\sim}{D}(R)$ operate in a vector space L_n. An equivalent representation ~~is~~ $\underset{\sim}{D}'(R)$ by any co-ordinate transformation

$$\underset{\sim}{D}'(R) = \underset{\sim}{C}\, \underset{\sim}{D}(R)\, \underset{\sim}{C}^{-1} \qquad (\underset{\sim}{C} \text{ non-singular})$$

- It is useful to have quantities which "characterize" a representation independent of the choice of co-ordinates. One such quantity is the trace

$$\sum_i D'_{ii}(R) = \sum_i \sum_{jk} C_{ij} D_{jk} (C^{-1})_{ki}$$

$$= \sum_{jk} \underbrace{\sum_i (C^{-1})_{ki} C_{ij}}_{\delta_{jk}} D_{jk} = \sum_j D_{jj}(R) = \chi(R)$$

$\chi(R)$ is called the character of $D(R)$. It is the same for equivalent representations, but has different values for different representations.

$\chi^{(\mu)}(R)$ is the character of R in the μ'th representation.

- Recall that conjugate classes of elements are obtained from one another by a similarity transformation. ie if R and S are in the same class, then another element U exists such that

$$S = URU^{-1}$$

$$\Rightarrow \underset{\sim}{D}(S) = \underset{\sim}{D}(U)\, \underset{\sim}{D}(R)\, \underset{\sim}{D}(U^{-1})$$

- It follows that in any given representation, elements in the same class have the same character.

- If a group has ν classes, then we can associate with the μ'th representation the set of characters

$$\chi_1^\mu, \chi_2^\mu \cdots \chi_\nu^\mu .$$