3 Interaction Between Charges - The Breit Equation.

3.1. Review of Non-Relativistic Scattering in the Interaction Picture. (see Taylor, "Scattering Theory" Section 9e)

- If $H = H_0 + V$, where V is responsible for scattering, then make the transformation

$$|\psi_t\rangle_I = e^{iH_0 t} |\psi_t\rangle_S$$

where I = interaction picture
S = Schrödinger picture.

The operators transform similarly according to

$$A_I(t) = e^{iH_0 t} A_S(t) e^{-iH_0 t}$$

so that $_I\langle \phi | A_I | \psi \rangle_I = {}_S\langle \phi | A_S | \psi \rangle_S$.

- The time evolution operator in the Schrödinger picture is

$$U_S(t, t_0) = e^{-iH(t-t_0)}$$

Therefore in the interaction picture

$$U_I(t, t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}$$

- The S-operator in the interaction picture is the limit

$$S = U_I(\infty, -\infty)$$

ie $|\psi_\infty\rangle_I = S |\psi_{-\infty}\rangle_I$

where $|\psi_{\pm\infty}\rangle_I$ correspond to the asymptotic "in" and "out" states in the Schrödinger picture.

<u>Perturbation Expansion for $U_I(t,t_0)$</u>

- $U_I(t,t_0)$ satisfies the equation

$$i\frac{d}{dt} U_I(t,t_0) = V_I(t)\, U_I(t,t_0)$$

where $V_I(t) = e^{iH_0t}\, V(t)\, e^{-iH_0t}$.

The above equation can be integrated to read

$$U_I(t,t_0) = 1 - i\int_{t_0}^{t} d\tau\, V_I(\tau)\, U_I(\tau,t_0)$$

The lower limit of integration is chosen so that $U_I(t_0,t_0) = 1$.

- If the interaction is weak, we can try an iterative solution, starting with $U_I(t,t_0)=1$.

$$U_I(t,t_0) = 1 - i\int_{t_0}^{t} d\tau\, V_I(\tau) + (-i)^2\int_{t_0}^{t} d\tau' \int_{t_0}^{\tau'} d\tau\, V_I(\tau')\, V_I(\tau) + \cdots$$

- Taking the limit $t\to\infty$, $t_0\to-\infty$ and using

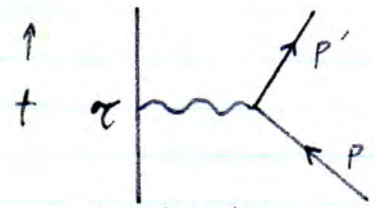
$$\langle \vec{p}\,'|V_I|\vec{p}\,\rangle = \langle\vec{p}\,'|e^{iE_{p'}t}\, V\, e^{-iE_pt}|\vec{p}\,\rangle$$

it follows that

$$\langle\vec{p}\,'|S|\vec{p}\,\rangle = \delta^{(3)}(\vec{p}\,'-\vec{p}\,) - i\int_{-\infty}^{\infty} d\tau \langle\vec{p}\,'|e^{iE_{p'}\tau}\, V e^{-iE_p\tau}|\vec{p}\,\rangle$$
$$+ (-i)^2\int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\tau'} d\tau \langle\vec{p}\,'|e^{iE_{p'}\tau'}\, V e^{-iH_0(\tau'-\tau)}\, V \times e^{-iE_p\tau}|\vec{p}\,\rangle + \cdots$$
$$= \delta^{(3)}(\vec{p}\,'-\vec{p}\,) + \langle\vec{p}\,'|S^{(1)}|\vec{p}\,\rangle + \langle\vec{p}\,'|S^{(2)}|\vec{p}\,\rangle + \cdots.$$

The individual terms are

1. $$\langle \vec{p}\,' | S^{(1)} | \vec{p} \rangle = -i \int_{-\infty}^{\infty} d\tau \, e^{i(E_{p'} - E_p)\tau} \langle \vec{p}\,' | V | \vec{p} \rangle$$
$$= -2\pi i \, \delta(E_{p'} - E_p) \langle \vec{p}\,' | V | \vec{p} \rangle$$



Fixed centre of force.

final state: $\langle \vec{p}\,' | e^{iE_{p'}\tau}$
interaction: $-iV$
initial state: $e^{-iE_p\tau} | \vec{p} \rangle$

Then integrate over all τ.

2. $\langle \vec{p}\,' | S^{(2)} | \vec{p} \rangle$
$$= (-i)^2 \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\tau'} d\tau \langle \vec{p}\,' | e^{iE_{p'}\tau'} V e^{-iH_0(\tau'-\tau)} V e^{-iE_p\tau} | \vec{p} \rangle$$

$$= (-i)^2 \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\tau'} d\tau \int d^3\vec{k} \, e^{iE_{p'}\tau'} \langle \vec{p}\,' | V | \vec{k} \rangle e^{-iE_k(\tau'-\tau)}$$
$$\times \langle \vec{k} | V | \vec{p} \rangle e^{-iE_p\tau}$$

In order to evaluate the integral over $d\tau$, insert a damping factor $\lim_{\epsilon \to 0+} e^{\epsilon\tau}$. Then
$$\int_{-\infty}^{\tau'} d\tau \, e^{-i(E_p - E_k + i\epsilon)\tau} = \frac{i \, e^{i(E_k - E_p)\tau'}}{E_p - E_k + i\epsilon}$$

The remaining integral over $d\tau'$ is
$$\int_{-\infty}^{\infty} d\tau' e^{i(E_{p'} - E_k + E_k - E_p)\tau'} = 2\pi\delta(E_{p'} - E_p)$$

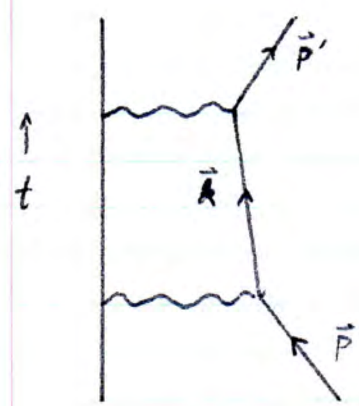
Thus $\langle \vec{p}\,' | S^{(2)} | \vec{p} \rangle = -2\pi i \, \delta(E_{p'} - E_p)$
$$\times \int d^3\vec{k} \, \frac{\langle \vec{p}\,' | V | \vec{k} \rangle \langle \vec{k} | V | \vec{p} \rangle}{E_p - E_k + i\epsilon}$$

Define $G^0(E_p+i0) = \int d^3\vec{k} \dfrac{|\vec{k}\rangle\langle\vec{k}|}{E_p - E_k + i0}$

$$= \frac{1}{E_p - H_0 + i0}$$

This is the Green's operator or free particle propagator. Then

$$\langle\vec{p}'|S^{(2)}|\vec{p}\rangle = -2\pi i\,\delta(E_{p'}-E_p)\langle\vec{p}'|V\,G^0(E_p+i0)V|\vec{p}\rangle$$



additional rule: insert $G^0(E_p+i0)$ for each internal electron line.

It is convenient to separate the factor $-2\pi i\,\delta(E_{p'}-E_p)$ and write

$$\langle\vec{p}'|S^{(n)}|\vec{p}\rangle = -2\pi i\,\delta(E_{p'}-E_p)\,\mathcal{U}^{(n)}_{\vec{p}',\vec{p}}$$

Then $\mathcal{U}^{(n)}_{\vec{p}',\vec{p}}$ can be interpreted as the

n th. order <u>interaction energy</u> producing the scattering. For example,

$$\mathcal{U}^{(1)}_{\vec{p}',\vec{p}} = \langle\vec{p}'|V|\vec{p}\rangle$$

3.2 Application to QED

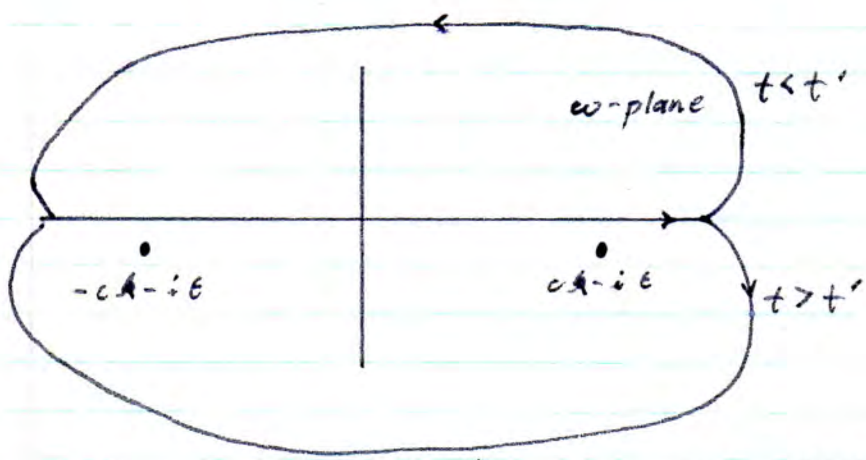
- The application to interacting systems of particles and fields is similar in principle. The main differences are
1. Electromagnetic interactions are not instantaneous. They are propagated by photons. There is no such thing as a local interaction "potential" (except as a lowest approximation). This is expressed mathematically by replacing internal photon lines by the photon propagator (Green's function).

The photon ~~propagator~~ Green's function is discussed by Jackson (Classical Electrodynamics, p. 185). The retarded solution to

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x},t;\vec{x}',t') = -4\pi\,\delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

is

$$G_R(\vec{x},t;\vec{x}',t') = \frac{1}{4\pi^3}\int d^3\vec{k}\int_{-\infty}^{\infty} d\omega\,\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')-i\omega(t-t')}}{\vec{k}^2 - \frac{1}{c^2}(\omega+i\epsilon)^2}$$



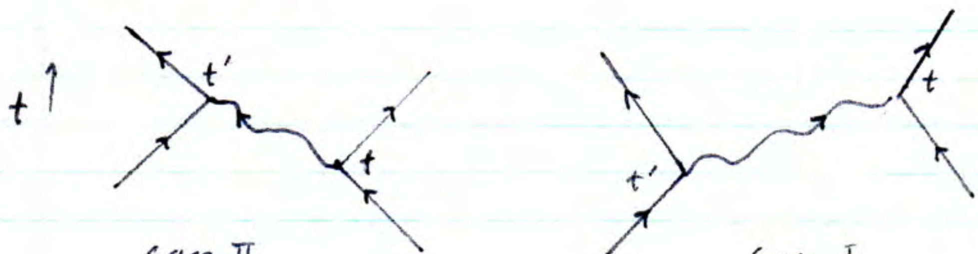
$$\text{denominator} = \left[\,|\vec{k}| - \frac{(\omega+i\epsilon)}{c}\right]\left[\,|\vec{k}| + \frac{(\omega+i\epsilon)}{c}\right]$$

This solution has the properties
$G_R = 0$ for $t < t'$

and G_R is a standing wave solution for $t > t'$. This can be seen by performing the integration over ω, with the result

$$G_R = \frac{c}{2\pi^2}\int d^3\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\,\frac{\sin[c|\vec{k}|(t-t')]}{|\vec{k}|}$$

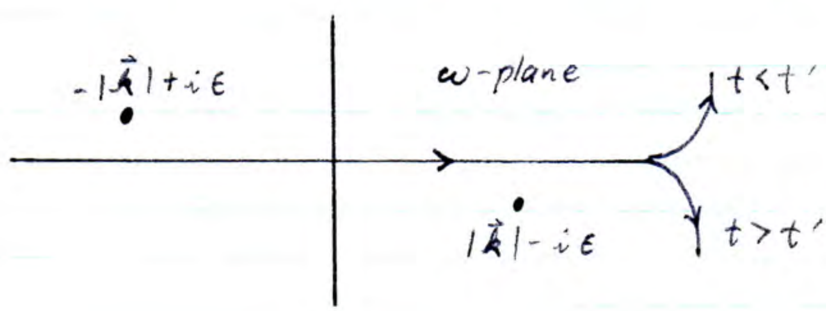
However, we want travelling wave solutions that reverse direction for $t<t'$ and $t>t'$.



Case II Case I

This is accomplished by defining

$$G^c(\vec{x},t;\vec{x}',t') = \frac{1}{4\pi^3}\int d^3\vec{k}\int d\omega\,\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')-i\omega(t-t')}}{\vec{k}^2-\frac{\omega^2}{c^2}-i\epsilon}$$



$$\text{denominator} = \left[(\vec{k}^2-i\epsilon)^{1/2}-\frac{\omega}{c}\right]\left[(\vec{k}^2-i\epsilon)^{1/2}+\frac{\omega}{c}\right]$$

$$\simeq\left[|\vec{k}|-\frac{i\epsilon}{2|\vec{k}|^2}-\frac{\omega}{c}\right]\left[|\vec{k}|-\frac{i\epsilon}{2|\vec{k}|^2}+\frac{\omega}{c}\right]$$

Case I: $t > t'$

$$G^c(\vec{x},t;\vec{x}',t') = \frac{2\pi i c}{4\pi^3}\int d^3\vec{k}\,\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}') - i|\vec{k}|c(t-t')}}{2|\vec{k}|}$$

Represents a wave travelling outward from $\vec{r}'$ to $\vec{r}$ with velocity c.

Case II: $t < t'$

$$G^c(\vec{x},t;\vec{x}',t') = \frac{2\pi i c}{4\pi^3}\int d^3\vec{k}\,\frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}') + i|\vec{k}|c(t-t')}}{2|\vec{k}|}$$

Represents a wave travelling inward with respect to case I.

~~Notation~~ Notation - Put $G^c(\vec{x},t;\vec{x}',t')$

$$= G^c(\vec{x}-\vec{x}', t-t') = G^c(x-x')$$

and note that

$$kx = \vec{k}\cdot\vec{r} - \frac{\omega}{c}t$$

$$k^2 = \vec{k}\cdot\vec{k} - \frac{\omega^2}{c^2}$$

$$d^4k = d^3\vec{k}\,\frac{d\omega}{c}$$

Then

$$G^c(x) = \frac{1}{4\pi^3}\int d^4k\,\frac{e^{ikx}}{k^2 - i\epsilon}$$

Note that A. and B. use the definition

$$D^c(x) = \frac{-i}{4\pi}\,G^c(x)$$

$$= \frac{-i}{(2\pi)^4}\int d^4k\,\frac{e^{ikx}}{k^2 - i\epsilon}$$

The quantity $D^c(k) = \frac{-i}{k^2 - i\epsilon}$

is the photon propagator in momentum space, such that

$$D^c(x) = \frac{1}{(2\pi)^4} \int d^4k \, D^c(k) e^{ikx}$$

2. The other main difference is that the electron propagator for internal electron lines

$$G^0(E_p + i0) = \frac{1}{E_p - H_0 + i0} \quad , \quad E_p = \frac{p^2}{2m}$$

must be replaced by its relativistic counterpart (see eq. 6)

$$S^c(p) = \frac{-i}{i\hat{p} + m} \quad \text{with } m = m_r - i0$$

ie. throughout, we regard the mass as a complex quantity with negative imaginary part in evaluating contour integrals. Multiply numerator and denominator by $-(i\hat{p} - m)$ to obtain

$$S^c_{\alpha\beta}(p) = \frac{i(i\hat{p} - m)_{\alpha\beta}}{p^2 + m^2}$$

ie. $S^c(p)$ is a 4×4 matrix operator with components labelled by α, β.

- The co-ordinate representation is

$$S^c_{\alpha\beta}(x) = \frac{1}{(2\pi)^4} \int d^4p \, S^c_{\alpha\beta}(p) e^{ipx}$$

This is the Green's function of the Dirac equation

$$(i\hat{p}+m)\,S^c(x) = i\,\delta^{(4)}(x)$$

The solution can be written in the form of a spectral decomposition

$$S^c_{\alpha\beta}(x_1,x_2) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t_1-t_2)} \sum_n \frac{\Psi_{n,\alpha}(\vec{r}_1)\bar{\Psi}_{n,\beta}(\vec{r}_2)}{E_n(1-i0)+\omega}$$

where the sum over n includes both positive and negative frequency solutions. It also includes an integration over continuum states.

- The above form is particularly useful for problems involving an electron in an external field - such as the Coulomb field of the nucleus; i.e. bound state problems. The external field is then clearly too large to be treated as a perturbation and must be included exactly.