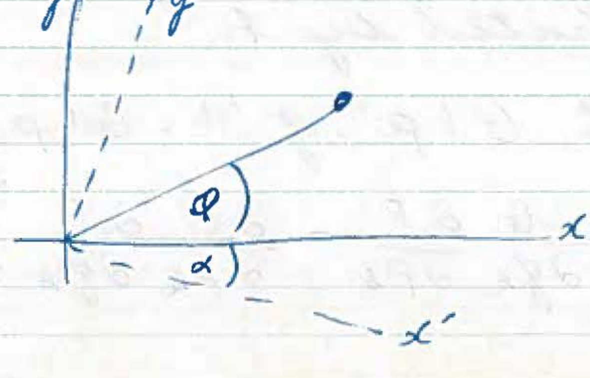
3. Every element a has an inverse a^{-1} $\quad aa^{-1} = a^{-1}a = 1$
4. Every product $c = ab$ is a member of the group.

Typically $ab \neq ba$.

- A subgroup is a subset of the group which also satisfies the group postulates within itself.

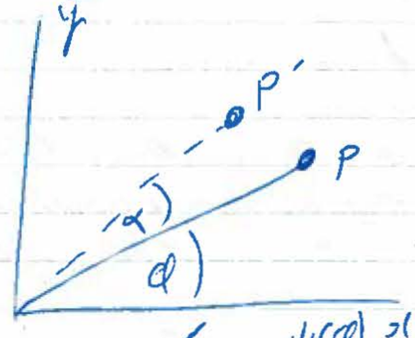
Examples
1. The set of all non-singular square matrices - full linear group $GL(n)$
2. The set of all unitary matrices forms a subgroup $U(n)$
3. The subset of 2 with det +1 is $SU(n)$
4. The set of all real linear transforms in n-dimensional space which preserve distances - ie rotations + reflections = all $n \times n$ orthogonal matrices $O(n)$

Rotation convention.
- Adopt convention of Brink + Satchler that co-ord. axes are rotated while system is held fixed



$$\psi(\phi) = \psi'(\phi+\alpha) \text{ or } \psi'(\phi) = \psi(\phi-\alpha)$$

Alternate system



$$\psi(\phi) = \psi'(\phi+\alpha)$$

$\psi'(\phi)$ is obtained from $\psi(\phi)$ by replacing ϕ by $\phi-\alpha$ ~~in $\psi(\phi)$~~

$\alpha > 0$ corresponds to positive (counterclockwise) rotation of system or negative rotation of axes.

Rotation Operators

- In classical mechanics, the linear and angular momenta are the generating fcns for translations and rotations.

If F is any fcn of p_i, q_i, then

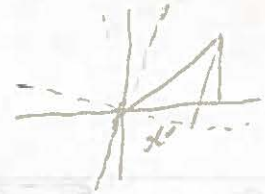
$$p_i' = p_i + \overset{\epsilon}{\alpha}\{p_i, F\}$$

$$q_i' = q_i + \overset{\epsilon}{\alpha}\{q_i, F\}, \quad \alpha \text{ small}$$

gives the canonical positions and momenta after the infinitesimal transformation ~~rotation~~ generated by F.

In general, $G(p_i', q_i') = G(p_i, q_i) + \alpha\{G, F\}$

$$\{G, F\} = \sum_k \left(\frac{\partial G}{\partial q_k}\frac{\partial F}{\partial p_k} - \frac{\partial G}{\partial p_k}\frac{\partial F}{\partial q_k} \right)$$



Put $F = L_z = x p_y - y p_x$

Then
$$x' = x - \overset{\epsilon}{\alpha} y \qquad p_x' = p_x - \overset{\epsilon}{\alpha} p_y$$
$$y' = y + \alpha x \qquad p_y' = p_y + \alpha p_x$$
$$z' = z \qquad p_z' = p_z$$

In addition, $\dfrac{dF}{dt} = \{F, H\} \Rightarrow \dfrac{dL_z}{dt} = \{L_z, H\}$

If rotation is a symmetry operation, then $H(\vec{r}\,', \vec{p}\,') = H(\vec{r}, \vec{p})$

But $H(\vec{r}\,', \vec{p}\,') = H(\vec{r}, \vec{p}) + \overset{\epsilon}{\alpha}\{H, L_z\}$

$\therefore \{H, L_z\} = 0$

and $\dfrac{dL_z}{dt} = 0$.

In q.m. the P.B. is replaced by the commutator.

ie $\{F, G\} \longrightarrow -\dfrac{i}{\hbar}[F, G]$
$$= -\frac{i}{\hbar}(FG - GF)$$

A q.m. operator A then transforms to

$$A' = A + \overset{\epsilon}{\alpha}\{A, L_z\}$$
$$\rightarrow A - \frac{i\alpha}{\hbar}(AL_z - L_zA)$$
$$= \left(1 + \frac{i\alpha}{\hbar}L_z\right)A\left(1 - \frac{i\alpha}{\hbar}L_z\right)$$
$$= D_\alpha^\dagger A D_\alpha \quad , \quad D_\alpha = 1 - \frac{i\overset{\epsilon}{\alpha}}{\hbar}L_z$$

A rotation through finite angle θ can be made in N small steps

$$D_\theta = \lim_{N \to \infty} \left(1 - \frac{i\theta}{N\hbar} L_z \right)^N$$

put $y = \left(1 - \frac{a}{N} \right)^N$

$$\ln y = N \ln\left(1 - \frac{a}{N}\right) \approx N\left\{\cancel{1} - \frac{a}{N} + \cdots\right]$$

$$= \frac{\ln(1 - a/N)}{1/N} \longrightarrow -a.$$

$$\longrightarrow \frac{1}{1 - a/N}\left(\frac{a}{N^2}\right)(-N^2) \longrightarrow -a$$

$$\therefore \lim_{N \to \infty} y = e^{-a}$$

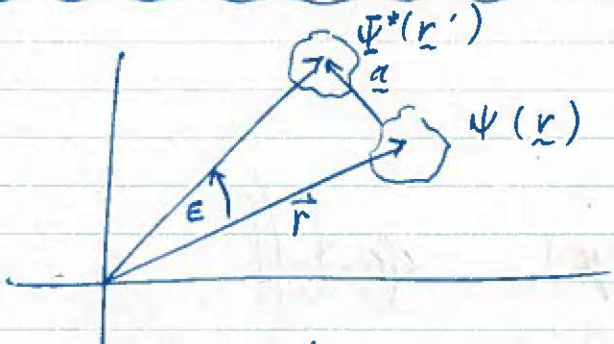
and $D_\theta = e^{-i\theta L_z/\hbar}$

- If a system has internal degrees of freedom, then it is the total angular mom. that is conserved, not just $\underset{\sim}{L}$.

$$\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$$

$$D_\theta = e^{-i\theta \underset{\sim}{J}\cdot\underset{\sim}{n}/\hbar}.$$

Alternative Derivation of Rotation Operator



$$\underset{\sim}{r}' = \underset{\sim}{r} + \underset{\sim}{a}$$

$$\Psi^*(\underset{\sim}{r}') = \psi(\underset{\sim}{r})$$

$$\Psi^*(\underset{\sim}{r}) = \psi(\underset{\sim}{r} - \underset{\sim}{a})$$

$$= \psi(\underset{\sim}{r}) - \underset{\sim}{a} \cdot \nabla \psi(\underset{\sim}{r}) + \cdots$$

$$\underset{\sim}{a} = \epsilon \hat{n} \times \underset{\sim}{r}$$

$$\therefore \delta \psi(\underset{\sim}{r}) = \Psi^*(\underset{\sim}{r}) - \psi(\underset{\sim}{r}) = -\epsilon \hat{n} \cdot (\nabla \times \underset{\sim}{r}) \psi(\underset{\sim}{r})$$

$$= -\frac{i}{\hbar} \epsilon \hat{n} \cdot \underset{\sim}{L} \, \psi(\underset{\sim}{r})$$

$$\therefore D_\theta = 1 - \frac{i}{\hbar} \epsilon \hat{n} \cdot \underset{\sim}{L}$$

Definition of a function of an operator.

Suppose $F(H) = a_0 + a_1 H + a_2 HH + \cdots$

and $H\phi_n = E_n \phi_n$, $HH\phi_n = E_n H \phi_n = E_n^2 \phi_n$ etc.

Then $F(H)\phi_n = F(E_n)\phi_n$

For a general fcn $\psi = \sum_n c_n \phi_n$

$$F(H)\psi = \sum_n c_n F(E_n)\phi_n.$$

Example - rotation about z-axis $\hat{n} = \hat{k}$

$$D_\theta = e^{-i\theta L_z/\hbar}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\ z$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\ (x+iy) \qquad \langle D_\theta \rangle = \Big($$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\ (x-iy)$$

$$\langle Y_1^0 | D_\theta | Y_1^0 \rangle = 1$$

$$\langle Y_1^{\pm 1} | D_\theta | Y_1^{\pm 1} \rangle = e^{\mp i\theta/\hbar}$$

$$D_\theta^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}$$

$$\sqrt{\frac{3}{4\pi}}\frac{x}{r} = \cancel{\frac{1}{2}\sqrt{\frac{8\pi}{3}}}\left(Y_1^{-1} - Y_1^1\right)/\sqrt{2}$$

$$\sqrt{\frac{3}{4\pi}}\frac{y}{r} = \cancel{-\frac{1}{2i}\sqrt{\frac{8\pi}{3}}}\left(Y_1^{-1} + Y_1^1\right)/\sqrt{2}\,i$$

$$\sqrt{\frac{3}{4\pi}}\frac{z}{r} = \cancel{\sqrt{\frac{4\pi}{3}}}\ Y_1^0$$

$$D_\theta^{(1)} = \begin{pmatrix} \frac{1}{2}(e^{i\theta} + e^{-i\theta}) & -\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) & 0 \\ \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) & \frac{1}{2}(e^{i\theta} + e^{-i\theta}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$