Coupling of Two Angular Momenta.

- Suppose the total angular momentum J of a system is composed of two parts J_1 and J_2. eg. two electrons, or spin-orbit for one electron.

- Neglect interaction so that both components are quantized separately.

$H, J_1^2, J_{1z}\ J_2^2, J_{2z}$ - form a complete set of commuting operators.

The eigenfunctions $|\alpha j_1 j_2 m_1 m_2\rangle$ can be written in a simple product form

$$|\alpha j_1 j_2 m_1 m_2\rangle = |\beta j_1 m_1\rangle |\gamma j_2 m_2\rangle$$

Then

$$J_1^2 |j_1 j_2 m_1 m_2\rangle = j_1(j_1+1)|j_1 j_2 m_1 m_2\rangle$$

$$J_{1z} |j_1 j_2 m_1 m_2\rangle = m_1 |j_1 j_2 m_1 m_2\rangle$$

Problem - find linear combinations of $|j_1 j_2 m_1 m_2\rangle$ with different $m_1 m_2$ such that

$$J^2 |j_1 j_2 J M\rangle = J(J+1)|j_1 j_2 J M\rangle$$

$$J_z |j_1 j_2 J M\rangle = M |j_1 j_2 J M\rangle$$

$$J^2 = (\underset{\sim}{J}_1 + \underset{\sim}{J}_2)^2 = J_1^2 + J_2^2 + 2\,\underset{\sim}{J}_1 \cdot \underline{J}_2$$

- If components are interacting, then it is only J^2 that is conserved and not J_1^2 and J_2^2 separately.

- Expansion coefs. are

$$|j_1 j_2 J M\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle\langle j_1 j_2 m_1 m_2 | J M\rangle$$

$\langle m_1 m_2 | J M\rangle$ is a unitary transformation matrix

$m_1 m_2$ - label rows } same no.
$J\,M$ - label columns. } of each.

Since matrix is unitary,

$$\sum_{m_1 m_2} \langle m_1 m_2 | J M\rangle^{*} \langle m_1 m_2 | J' M'\rangle$$

$$= \sum_{m_1 m_2} \langle J M | m_1 m_2\rangle\langle m_1 m_2 | J' M'\rangle = \delta(JJ')\delta(mm')$$

Similarly

$$\sum_{JM} \langle m_1 m_2 | J M\rangle\langle J M | m_1' m_2'\rangle = \delta(m_1 m_1')\delta(m_2 m_2')$$

<u>Selection Rules.</u>

1) $J_z |j_1 j_2 J M\rangle = M |j_1 j_2 J M\rangle$

$$= (J_{1z} + J_{2z}) \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle\langle m_1 m_2 | J M\rangle$$

$$= \sum_{m_1 m_2} (m_1 + m_2) |j_1 j_2 m_1 m_2\rangle\langle m_1 m_2 | J M\rangle$$

$\therefore\ M = m_1 + m_2$ if $\langle m_1 m_2 | J M\rangle \neq 0.$

2) triangular rule.

consider $\langle \overset{m_1=}{j_1}\ \overset{m_2=}{J - j_1} | J\, J\rangle$

$$-j_2 \leq J - j_1 \leq j_2 \Rightarrow (j_1 - j_2) \leq J \leq (j_1 + j_2)$$

similarly consider $\langle J - j_2\ \ j_2 | J\, J\rangle$

$$-j_1 \leq J - j_2 \leq j_1 \Rightarrow (j_2 - j_1) \leq J \leq (j_1 + j_2)$$

$\therefore \ |j_1 - j_2| \leq J \leq j_1 + j_2 \quad \text{if } \langle m_1 m_2 | J M \rangle \neq 0.$

Group Theory Interpretation of Vector addition.

The simple products of states $|j_1 m_1\rangle |j_2 m_2\rangle$ span a $(2j_1+1)(2j_2+1)$ dimensional space.

Under rotations

$$|j n\rangle' = \sum_m \mathcal{D}^{j}_{mn}(R)\,|j m\rangle$$

$$\therefore \ |j_1 j_2 n_1 n_2\rangle = \sum_{m_1 m_2} \mathcal{D}^{j_1}_{m_1 n_1}(R)\,\mathcal{D}^{j_2}_{m_2 n_2}(R)\,|j_1 j_2 m_1 m_2\rangle$$

The matrix elements of $\mathcal{D}^{j_1}(R) \times \mathcal{D}^{j_2}(R)$ form a reducible $(2j_1+1)(2j_2+1)$ dimensional representation of the rotation group.

- The Clebsch-Gordan ~~basis~~ unitary transformation to the $|j_1 j_2 J M\rangle$ basis set brings matrices to block diagonal form

$$\mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2} = \begin{pmatrix} \boxed{\mathcal{D}^{J_1}} & & 0 \\ & \boxed{\mathcal{D}^{J_2}} & \\ 0 & & \boxed{\mathcal{D}^{J_3}} \end{pmatrix}$$

dimensions run from $j_1 + j_2$ to $|j_1 - j_2|$.

$$\mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2} = \sum_{J=|j_1-j_2|}^{j_1+j_2} \mathcal{D}^{J}$$

Clebsch-Gordan series.

Explicitly,

$$\langle j_1 m_1 | \langle j_2 m_2 | \mathcal{D}^{j_1} \otimes \mathcal{D}^{j_2} | j_1 n_1 \rangle | j_2 n_2 \rangle$$

$$= \sum_J \langle j_1 j_2 m_1 m_2 | \mathcal{D}^J | j_1 j_2 n_1 n_2 \rangle$$

ie $\mathcal{D}^{j_1}_{m_1 n_1} \mathcal{D}^{j_2}_{m_2 n_2} = \sum_{JMN} \langle j_1 j_2 m_1 m_2 | J M \rangle \langle J M | \mathcal{D}^J | J N \rangle$
$\langle J N | j_1 j_2 n_1 n_2 \rangle$

$$\mathcal{D}^{j_1}_{m_1 n_1} \mathcal{D}^{j_2}_{m_2 n_2} = \sum_{JMN} \langle m_1 m_2 | J M \rangle \mathcal{D}^J_{MN} \langle J N | n_1 n_2 \rangle$$

The inverse transform is

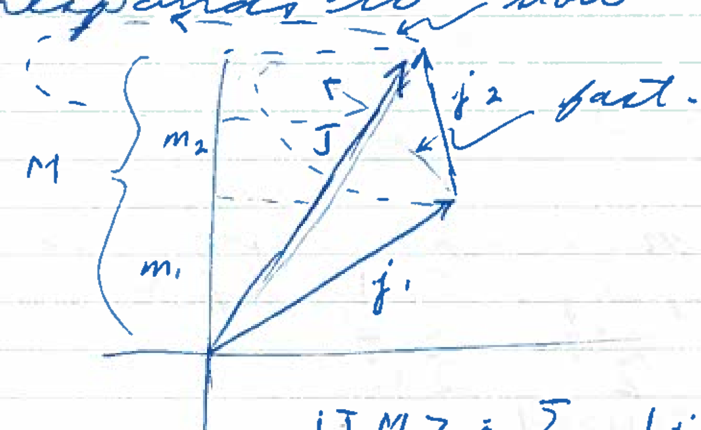
$$\mathcal{D}^J_{MN} = \langle J M | D | J N \rangle$$

$$= \sum_{\substack{m_1 m_2 \\ n_1 n_2}} \langle J M | m_1 m_2 \rangle \langle m_1 m_2 | D | n_1 n_2 \rangle \langle n_1 n_2 | J N \rangle$$

$$= \sum_{\substack{m_1 m_2 \\ n_1 n_2}} \langle J M | m_1 m_2 \rangle \mathcal{D}^{j_1}_{m_1 n_1} \mathcal{D}^{j_2}_{m_2 n_2} \langle n_1 n_2 | J N \rangle$$

Vector Model

- Physically, the vector coupling corresponds to slow



m_1 & m_2 fluctuate as j_1 and j_2 precess about J.

$$| J M \rangle = \sum_{m_1 m_2} | j_1 j_2 m_1 m_2 \rangle \langle m_1 m_2 | J M \rangle$$

$|\langle m_1 m_2 | J M \rangle|^2$ = prob. of measurement of J_{1z} yielding m_1 ($m_2 = M - m_1$)

The matrix representation of $\underset{\sim}{D}^{j_1}(R) \otimes \underset{\sim}{D}^{j_2}(R)$ looks like

$$\begin{array}{c} (2j_1+1) \longrightarrow \\ \left(\begin{array}{ccc} D^{j_1}_{11}(R)\left(\underset{\sim}{D}^{j_2}(R)\right) & D^{j_1}_{12}(R)\left(\underset{\sim}{D}^{j_2}(R)\right) & \cdots \\ \vdots & & \\ & & \end{array} \right) \end{array}$$

All matrix elements are in general non-zero. The Clebsch-Gordan transformation brings the matrix to block diagonal form, with each block representing an invariant subspace.

Since the coupled states $|j_1 j_2 J M\rangle$ and the operators J^2, J_z satisfy the same angular momentum algebra as $|j_1 m_1\rangle$, j_1^2, j_{1z}, states with the same J and different M merely transform amongst themselves under rotations, and therefore form a basis for an irreducible representation.

Proof of Symmetry Property for V.C. Coefs.

- Choose the phases of the wave functions so that

$$\langle j_1 j_2 J M | j_{1z} | j_1 j_2 J{+}1\, M \rangle \quad \text{all } J \qquad (1)$$

is real and positive. Then

$$\langle j_1 j_2 J M | j_{2z} | j_1 j_2 J{+}1\, M \rangle \quad \text{all } J \qquad (2)$$

is real and negative since

$$\langle j_1 j_2 J M | J_z | j_1 j_2 J{+}1\, M \rangle = 0.$$

- The subscripts 1 and 2 label the dummy variables of integration for the two particles. They can therefore be interchanged in (2) to yield

$$\langle j_2 j_1 J M | j_{1z} | j_2 j_1 J{+}1\, M \rangle < 0 \qquad (3)$$

$\therefore$ the off-diagonal matrix elements of j_{1z} have opposite signs in the $(j_1 j_2) J M$ and the $(j_2 j_1) J M$ schemes.

- This implies that if $|j_1 j_2 J M\rangle = \pm |j_2 j_1 J M\rangle$

then $|j_1 j_2 J{+}1\, M\rangle = \mp |j_2 j_1 J{+}1\, M\rangle$

$\therefore$ in general $\underline{\underline{|j_1 j_2 J M\rangle = (-1)^{K+J} |j_2 j_1 J M\rangle}}$

The common phase factor $(-1)^K$ can be determined by considering the "stretched" state $J = j_1 + j_2$, $M = j_1 + j_2$, for which the two coupling schemes coincide

i.e. $|j_1 j_2, j_1{+}j_2, j_1{+}j_2\rangle = |j_1 j_1\rangle |j_2 j_2\rangle$

$= |j_2 j_2\rangle |j_1 j_1\rangle = |j_2 j_1, j_1{+}j_2, j_1{+}j_2\rangle$

i.e. $K = -(j_1 + j_2)$ and

$$|j_1 j_2 J M\rangle = (-1)^{J - j_1 - j_2} |j_2 j_1 J M\rangle.$$

Since $|j_1 j_2 J M\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | J M\rangle |j_1 m_1\rangle_1 |j_2 m_2\rangle$

and $|j_2 j_1 J M\rangle = \sum_{m_1 m_2} \langle j_2 j_1 m_2 m_1 | J M\rangle |j_2 m_2\rangle_2 |j_1 m_1\rangle$

$\therefore \langle j_1 j_2 m_1 m_2 | J M\rangle = (-1)^{J - j_1 - j_2} \langle j_2 j_1 m_2 m_1 | J M\rangle$

Example - for the $S = 1$, $M_S = 0$ spin state for two electrons,

$$\begin{aligned}
|\overset{j_1}{\tfrac{1}{2}}\ \overset{j_2}{\tfrac{1}{2}}\ 1\ 0\rangle &= \frac{1}{\sqrt{2}}\left(\alpha(1)\beta(2) - \beta(1)\alpha(2)\right) \\
&= -\frac{1}{\sqrt{2}}\left(\alpha(2)\beta(1) - \beta(2)\alpha(1)\right) \\
&= -|\overset{j_2}{\tfrac{1}{2}}\ \overset{j_1}{\tfrac{1}{2}}\ 1\ 0\rangle
\end{aligned}$$

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$= |j_2 j_2\rangle |j_1 j_1\rangle = |j_2 j_1, j_1+j_2, j_1+j_2\rangle$

$\therefore K = -(j_1 + j_2)$ and

$$|j_1 j_2 JM\rangle = (-1)^{J-j_1-j_2} |j_2 j_1 JM\rangle.$$

Since $|j_1 j_2 JM\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | JM\rangle |j_1 m_1\rangle_1 |j_2 m_2\rangle_2$

and $|j_2 j_1 JM\rangle = \sum_{m_1 m_2} \langle j_2 j_1 m_2 m_1 | JM\rangle |j_2 m_2\rangle_2 |j_1 m_1\rangle_1$

$$\therefore \langle j_1 j_2 m_1 m_2 | JM\rangle = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | JM\rangle$$

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$$= -\frac{1}{\sqrt{2}}\left(\alpha(2)\beta(1) - \beta(2)\alpha(1)\right)$$

$$= -|\overset{j_2}{\tfrac{1}{2}}\ \overset{j_1}{\tfrac{1}{2}}\ 1\ 0\rangle$$