Similarly the vector field operators

∇, $\underset{\sim}{L}$, $\nabla \times \underset{\sim}{L}$ are isotropic.

Note that for a given LM, there are 3 linearly independent vector spherical harmonics:- $\underset{\sim}{Y}^{M}_{L\,L+1\,1}$, $\underset{\sim}{Y}^{M}_{L\,L\,1}$, $\underset{\sim}{Y}^{M}_{L\,L-1\,1}$.

Instead of the above 3, we can construct instead

$$\nabla \phi_{LM}, \quad \underset{\sim}{L}\, \phi_{LM}, \quad \nabla \times \underset{\sim}{L}\, \phi_{LM}.$$

e.g. if $\phi_{LM} = Y_L^{M}$, then

$$\underset{\sim}{L}\, Y_L^{M} = \left[L(L+1) \right]^{1/2} \underset{\sim}{Y}^{M}_{L\,L\,1}.$$

$$\nabla Y_L^{M} = \frac{l}{r} \left[\frac{l+1}{2l+1} \right]^{1/2} \underset{\sim}{Y}^{M}_{L\,L+1\,1} + \frac{(l+1)}{r} \left[\frac{l}{2l+1} \right]^{1/2} Y^{M}_{L\,L-1\,1}$$

$$\nabla \times \underset{\sim}{L}\, Y_L^{M} = -i \frac{l^2}{r} \left[\frac{l+1}{2l+1} \right]^{1/2} \underset{\sim}{Y}^{M}_{L\,L+1\,1} + i \frac{(l+1)^2}{r} \left[\frac{l}{2l+1} \right]^{1/2} Y^{M}_{L\,L-1\,1}.$$

Quantities of the above type are convenient for expanding photon vector potential, which satisfies equation

$$(\nabla^2 + k^2)\, \vec{A} = 0.$$

Each vector component can be expanded in terms of the functions

$$\phi_{LM} = i^{L} (2L+1)\, j_L(kr)\, C_{LM}(\theta, \phi)$$

$$\text{ie. } e^{ikz} = \sum_L \phi_{L0}$$

Define

$$\underset{\sim}{A}_{LM} = \frac{1}{ik} \nabla \phi_{LM}$$

$$\underset{\sim}{A}^{e}_{LM} = \frac{1}{k\sqrt{L(L+1)}} \nabla \times L \phi_{LM}$$

$$\underset{\sim}{A}^{m}_{LM} = \frac{1}{\sqrt{L(L+1)}} L \phi_{LM}$$

Then clearly

$\nabla \times \underset{\sim}{A}_{LM} = 0$ irrotational

$\nabla \cdot \underset{\sim}{A}^{e}_{LM} = 0, \quad \nabla \cdot A^{m}_{LM} = 0$ solenoidal.

$\underset{\sim}{A}_{LM}$ and $\underset{\sim}{A}^{e}_{LM}$ have parity $(-1)^{L+1}$

A^{m}_{LM} " " $(-1)^{L}$

As we shall see, $\underset{\sim}{A}_{LM}$ forms longitudinal field components while A^{e}_{LM} + A^{m}_{LM} form transverse components. $\left.\begin{array}{l} A^{e}_{LM} = \frac{1}{k} \nabla \times A^{m}_{LM} \\ A^{m}_{LM} = 1/k\, \nabla \times A^{e}_{LM} \end{array}\right\}$ mutually $\perp$ and to $\underset{\sim}{A}_{LM}$.

$(\nabla^2 + k^2)\underset{\sim}{A} = 0$ has as solutions of form $\hat{e}_q\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$

Take $\underset{\sim}{k} = k_z \hat{k}$. Then

$\underset{\sim}{A} = \hat{e}_q e^{ikz}$ = wave propagating in

$\underset{\sim}{k} = k_z \hat{k}$ direction with polarization $\hat{e}_q$.

$q = \begin{array}{l} \pm 1 \\ 0 \end{array}$ $\begin{array}{l} \text{right + left circular polarization} \\ \text{longitudinal polarization.} \end{array}$

$$\hat{e}_q\, e^{ikz} = \sum_l \frac{i^l}{\sqrt{4\pi(2l+1)}}\, j_l(kr)\, Y_{l0}(\theta,\phi)\,\hat{e}_q\,.$$

$$= \sum_{Ll} \frac{i^l}{\sqrt{4\pi(2l+1)}}\, j_l(kr) \langle l\, 1\, 0\, q | L\, q\rangle\, \underset{\sim}{Y}^{q}_{Ll1}$$

Expansion of plane wave propagating in ~~the~~ $\underset{\sim}{k} = k_z\hat{k}$ direction with polarization $\hat{e}_q$ in terms of spherical waves with angular momentum L and definite parity.

The longitudinal and transverse components can be expressed simply in terms of the $\underset{\sim}{A}_{LM}$'s.

Longitudinal Component.

$$e^{ikz} = \sum_L \phi_{L0}\,, \qquad \phi_{LM} = i^L (2L+1)\, j_L(kr)\, C_{LM}(\theta,\phi)$$

$$\hat{e}_0\, e^{ikz} = \frac{1}{ik} \nabla e^{ikz}$$

$$= \frac{1}{ik} \sum_L \nabla \phi_{L0} = \sum_L \underset{\sim}{A}_{L0}$$

since $A_{LM} = \frac{1}{ik} \nabla \phi_{LM}$.

By similar arguments, one can show that the transverse components are

$$\hat{e}_q\, e^{ikz} = -\frac{1}{\sqrt{2}} \sum_L \left(q\, \underset{\sim}{A}^{(m)}_{Lq} + \underset{\sim}{A}^{(e)}_{Lq} \right) \qquad q = \pm 1$$

For a wave travelling in arbitrary direction $\underset{\sim}{k}$,

$$\hat{e}_q\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} = -\frac{1}{\sqrt{2}} \sum_{LM} \left(q A^{m}_{LM} + A^{e}_{LM} \right) \mathcal{D}^{L}_{Mq}(R)$$

R rotates z-axis into direction $\underset{\sim}{k}$. $\hat{e}_q$ refers to new co-ord. system.

The A_{LM} are the natural fields for expanding a longitudinal vector wave — eg sound wave.

The A^e_{LM} + A^m_{LM} correspond to the division of a transverse wave into electric and magnetic multipoles.

Spinor Fields

The general spin-½ field has two components.

$$\underset{\sim}{S} = \sum_\sigma \chi_\sigma S_\sigma(r,\theta,\phi)$$

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In this case, the generator of infinitesimal rotations is

$$\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$$

with $\underset{\sim}{S} = \frac{1}{2}\underset{\sim}{\sigma} = \frac{1}{2}(\sigma_x \hat{i} + \sigma_y \hat{j} + \sigma_z \hat{k})$

$\hat{L}$ rotates the scalar fields $S_\sigma(r,\theta,\phi)$ while $\hat{S}$ mixes the components.

$$\phi_{jln} = \sum_{\sigma m} \langle j n | \tfrac{1}{2}\, l\, \sigma\, m \rangle \chi_\sigma\, i^l\, C_{lm}(\theta,\phi)$$

is the analog of the vector spherical harmonic. The spinor plane wave can be expanded

$$\chi_\sigma e^{ikz} = \sum_{lj} (2l+1) \langle \tfrac{1}{2}\, l\, \sigma\, 0 | j\, \sigma \rangle j_l(kr)\, \phi_{jl\sigma} .$$

ϕ_{jln} has transformation properties of an irreducible tensor of rank $2j+1$.

Proof that $\hat{e}_q e^{ikz} = -\frac{1}{\sqrt{2}} \sum_L \left(q \vec{A}_{Lq}^{(m)} + \vec{A}_{Lq}^{(e)} \right)$. $q = \pm 1$

It is always possible to write

$$\hat{e}_q e^{ikz} = \sum_L \left(a_L \vec{A}_{Lq} + b_L \vec{A}_{Lq}^{(m)} + c_L \vec{A}_{Lq}^{(e)} \right)$$

1. Then $\nabla \cdot \hat{e}_q e^{ikz} = 0 = \sum_L a_L \nabla \cdot \vec{A}_{Lq}$

$$= \frac{1}{ik} \sum_L a_L \nabla^2 \phi_{Lq} = \frac{k}{i} \sum_L a_L \phi_{Lq}$$

$\therefore a_L = 0$.

2. $\nabla \times \hat{e}_q e^{ikz} = ik(\hat{k} \times \hat{e}_q) e^{ikz}$

But $\hat{k} \times \hat{e}_q = \mp \frac{1}{\sqrt{2}} \hat{k} \times (\hat{e}_x \pm i \hat{e}_y)$

$$= \mp \frac{1}{\sqrt{2}} (\hat{e}_y \mp i \hat{e}_x) = \frac{i}{\sqrt{2}} (\hat{e}_x \pm i \hat{e}_y)$$

$$= -q\, i\, \hat{e}_q$$

$\therefore \nabla \times \hat{e}_q e^{ikz} = kq\, \hat{e}_q e^{ikz}$

$$= \sum_L \left(b_L \nabla \times \vec{A}_{Lq}^{(m)} + c_L \nabla \times \vec{A}_{Lq}^{(e)} \right)$$

$$= k \sum_L \left(b_L \vec{A}_{Lq}^{(e)} + c_L \vec{A}_{Lq}^{(m)} \right)$$

$\therefore q b_L = c_L$ and $q c_L = b_L$.

3. Finally, $\hat{e}_q \cdot \vec{L} e^{ikz} = \hat{e}_q \cdot \vec{L} \sum_l \phi_{l0} = \mp \frac{1}{\sqrt{2}} L_\pm \sum_l \phi_{l0}$

$$= -\frac{q}{\sqrt{2}} \sum_l [l(l+1)]^{1/2} \phi_{lq}$$

The RHS is $\vec{L}\cdot\sum_\ell\left(b_\ell\vec{A}_{\ell q}^{(m)} + c_\ell\vec{A}_{\ell q}^{(e)}\right)$

$$= \sum_\ell\left\{b_\ell\,[\ell(\ell+1)]^{1/2}\,\phi_{\ell q} + c_\ell\cdot 0\right\}$$

$$\therefore\ b_\ell = -\frac{q}{\sqrt{2}},\qquad c_\ell = -\frac{1}{\sqrt{2}}.$$

The final result is thus

$$\hat{e}_q e^{ikz} = -\frac{1}{\sqrt{2}}\sum_\ell\left(q\,\vec{A}_{\ell q}^{(m)} + \vec{A}_{\ell q}^{(e)}\right)$$

Electromagnetic Multipoles

- The electric and magnetic fields E and H are related to the scalar and vector potentials by $\vec{H} = \nabla \times \vec{A}$, $\vec{E} = -\nabla V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$.

The gauge condition $\nabla \cdot \vec{A} = 0$ eliminates the longitudinal component $\nabla \phi_{LM}$ and $\vec{A}$ is of the form

$$\vec{A} = \vec{A}^{e} + \vec{A}^{m}$$

$$= \sum_{LM} \left(q_{LM}^{e} \vec{A}_{LM}^{e} + q_{LM}^{m} \vec{A}_{LM}^{m} \right)$$

In general, the $\vec{A}_{LM}^{e}$ and $\vec{A}_{LM}^{m}$ are complex. If the photon states are normalized to unity in a spherical box, then

$$\vec{A}^{e}(\vec{r}) = \sum_{LM} \left(\frac{\hbar\omega}{R}\right)^{1/2} \left(a_{LM}^{e} \vec{A}_{LM}^{e}(\vec{r}) + a_{LM}^{e+} A_{LM}^{e+}(r) \right)$$

a_{LM}^{e+} = creation operator for electric 2^L pole photon.

Multipole Sources

- Suppose we have a charge distribution $\rho(\underline{r}')$

$$V(\underline{r}) = \int \frac{\rho(r')}{|\underline{r}' - \underline{r}|} \, d\underline{r}'$$

$$\frac{1}{|\underline{r}' - \underline{r}|} = \sum_{lm} \frac{r'^{\,l}}{r^{l+1}} C_{lm}(\theta, \phi) C^*_{lm}(\theta', \phi') \qquad r > r'$$

$$\therefore \quad V(r) = \sum_{lm} Q^*_{lm} \frac{1}{r^{l+1}} C_{lm}(\theta, \phi)$$

~~But $\nabla^2 V = -4\pi \rho(\vec{r})$~~

where $Q_{lm} = \int \rho(\vec{r}') r'^{\,l} C_{lm}(\theta', \phi') \, d\vec{r}'$.

But $\nabla^2 V = -4\pi \rho(\vec{r})$

$$\therefore \sum_{lm} Q^*_{lm} \nabla^2 \left(\frac{1}{r^{l+1}} C_{lm}(\theta, \phi) \right) = -4\pi \rho(\vec{r})$$

Proof that $\langle J \| \vec{A} \| J \rangle = \dfrac{1}{\sqrt{J(J+1)}} \langle J M | \vec{J} \cdot \vec{A} | J M \rangle$

$$\langle J M | A_q | J' M' \rangle$$

$$\langle J M | \vec{J} \cdot \vec{A} | J M \rangle$$

$$= \sum_{J'M'q} \langle J M | J_q | J' M' \rangle \langle J' M' | A_{-q} | J M \rangle (-1)^q$$

$$= \sum_{M'q} \langle J \| \vec{J} \| J' \rangle \langle J M | J' 1 M' q \rangle \langle J M | J' 1 M' q \rangle \langle J \| \vec{A} \| J' \rangle$$

$$= \sqrt{J(J+1)} \, \langle J \| \vec{A} \| J \rangle$$

Matrix Elements of Tensor Operators.

The simplest example is the Wigner-Eckart theorem

$$\langle JM | T_{KQ} | J'M' \rangle = (-1)^{2K} \langle JM | J'KM'Q \rangle \langle J \| T_K \| J' \rangle$$

- Extend to cases where T_{KQ} or the states $|J'M'\rangle$ may be composite.

We first apply the W.E. theorem to prove the "Projection Theorem" for vector operators.

- What is matrix element of vector operator $\vec{A}$ between states with same J?

For any $\vec{A}$

$$\langle JM' | A^q | JM \rangle = \frac{\langle J \| \vec{A} \| J \rangle}{\langle J \| \vec{J} \| J \rangle} \langle JM' \, J^q | JM \rangle$$

$$\langle JJ | J_z | JJ \rangle = J = \langle JJ | J1J0 \rangle \langle J \| \vec{J} \| J \rangle$$

Since $\langle JJ | J1J0 \rangle = \sqrt{\frac{J}{J+1}}$

$$\langle J \| \vec{J} \| J \rangle = \sqrt{J(J+1)}$$

$$\therefore \langle JM | A^q | JM \rangle = \underline{\langle J \| \vec{A} \| J \rangle} \langle JM' \, \frac{J^q}{\sqrt{J(J+1)}} | JM \rangle$$

also $\langle J \| \vec{A} \| J \rangle = c \langle JM | \vec{J} \cdot \vec{A} | JM \rangle$

where c is independent of $\vec{A}$ and M.
Put $\vec{A} = \vec{J}$

$$\sqrt{J(J+1)} = c\, J(J+1)$$

$$\therefore c = \frac{1}{\sqrt{J(J+1)}}$$

Thus

$$\langle JM'|A^q|JM\rangle = \frac{\langle JM|\vec{J}\cdot\vec{A}|JM\rangle\langle JM'|J^q|JM\rangle}{J(J+1)}$$

But $\langle JM'|J^q|JM\rangle\langle JM|J\cdot A|JM\rangle$

$$= \sum_{M''}\langle JM'|J^q|JM''\rangle\langle JM''|J\cdot A|JM\rangle$$

$$= \langle JM'|J^q(\vec{J}\cdot\vec{A})|JM\rangle$$

∴ in general,

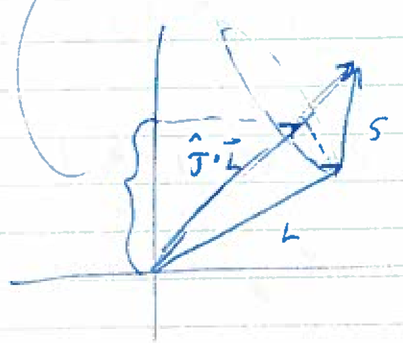
$$\langle JM'|\vec{A}|JM\rangle = \frac{\langle JM'|\vec{J}(\vec{J}\cdot\vec{A})|JM\rangle}{J(J+1)}$$

If $\hat{J} = \dfrac{\vec{J}}{\sqrt{J(J+1)}}$, then $\hat{J}(\hat{J}\cdot\vec{A})$ is just the projection of $\vec{A}$ in $\vec{J}$ direction.

e.g. suppose $\vec{J} = \vec{L}+\vec{S}$ and $A = L_z$

$$\langle JM|L_z|JM\rangle = \langle JM|\frac{J_z(\vec{J}\cdot\vec{L})}{J(J+1)}|JM\rangle$$

$$= \frac{M_J}{J}\langle JM|\frac{\vec{J}\cdot\vec{L}}{J+1}|JM\rangle$$

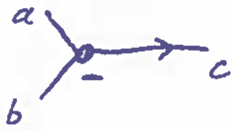


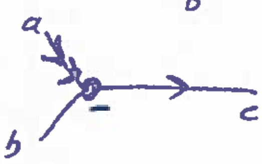
df $T_{KQ}(k_1, k_2) = \sum_{q_1 q_2} R_{k_1 q_1} S_{k_2 q_2} \langle k_1 k_2 q_1 q_2 | KQ \rangle$

then $\langle J \| T_K \| J' \rangle = (-1)^{2(K+k_1+k_2)} \sum_{\substack{q_1 q_2 M' M'' Q \\ J''}}$

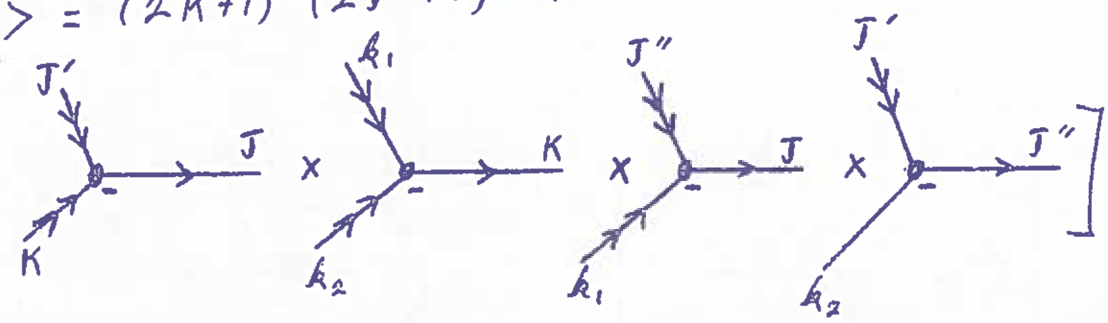
$\times \langle JM | J'KM'Q \rangle \langle KQ | k_1 k_2 q_1 q_2 \rangle \langle JM | J'' k_1 M'' q_1 \rangle$

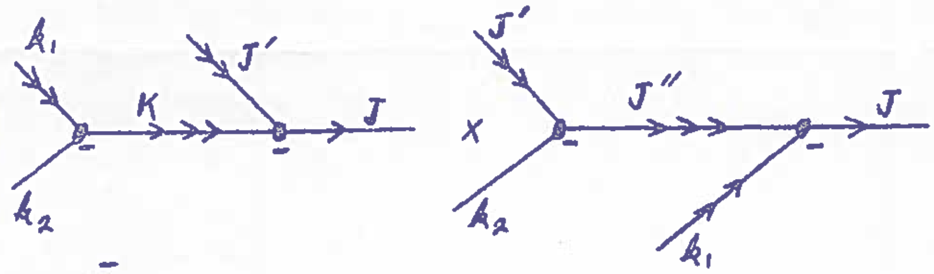
$\times \langle J''M'' | J' k_2 M'' q_2 \rangle \langle J \| R_{k_1} \| J'' \rangle \langle J'' | S_{k_2} \| J \rangle$ (1)

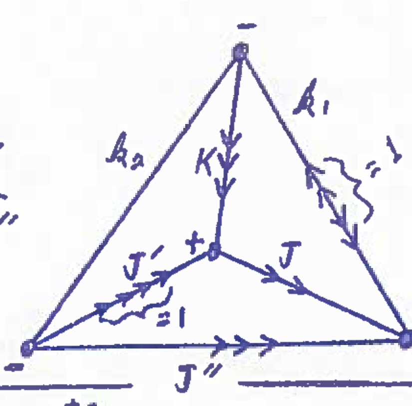
Use $\langle a b \alpha \beta | c \gamma \rangle = (-1)^{2a} (2c+1)^{1/2}$ 

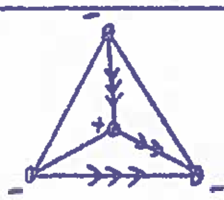
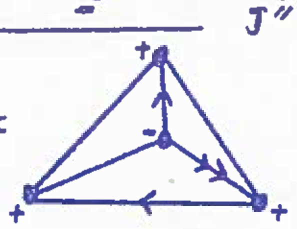
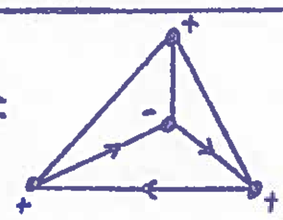
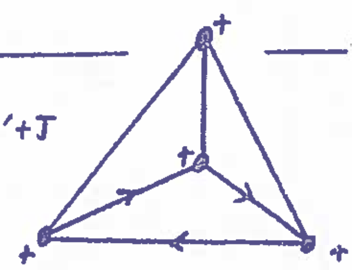
$= (2c+1)^{1/2}$ 

Then $\langle J \| T_K \| J' \rangle = (2K+1)^{1/2} (2J''+1)^{1/2} (2J+1)$

$\times \sum_{\substack{q_1 q_2 M M' M'' Q \\ J''}} \frac{1}{2J+1}$ []

$= (2K+1)^{1/2} (2J''+1)^{1/2} \sum_{\substack{q_1 q_2 M M' \\ J''}}$ []

$= (2K+1)^{1/2} (2J''+1)^{1/2}$ [$\sum_{J''}$  $\times \langle J \| R_{k_1} \| J'' \rangle \langle J'' \| S_{k_2} \| J \rangle$]

 =  =  $= (-1)^{K+J'+J}$ 

$= (-1)^{K+J'+J} \begin{Bmatrix} k_1 & k_2 & K \\ J' & J & J'' \end{Bmatrix}$

$\therefore \langle J \| T_K \| J' \rangle = \sum_{J''} (-1)^{K+J'+J} \begin{Bmatrix} k_1 & k_2 & K \\ J' & J & J'' \end{Bmatrix} (2K+1)^{1/2} (2J''+1)^{1/2}$

$\times \langle J \| R_{k_1} \| J'' \rangle \langle J'' \| S_{k_2} \| J' \rangle.$