Take inverse

$$Y_L^{M} Y_{l'}^{m'} = \sum_{kq} \langle kq | L l' M m' \rangle \langle L l' 0 0 | k 0 \rangle \times \left(\frac{(2L+1)(2l'+1)}{4\pi (2k+1)} \right)^{1/2} Y_k^{q}(\theta, \phi) .$$

$$\therefore \int Y_l^{m*} Y_L^{M} Y_{l'}^{m'} \, d\Omega$$

$$= \langle l m | L l' M m' \rangle \underbrace{\langle L l' 0 0 | k 0 \rangle \left(\frac{(2L+1)(2l'+1)}{4\pi (2k+1)} \right)^{1/2}}_{\langle l \| Y_L \| l' \rangle}$$

Read sections 4.8, 4.9.

Multipole Expansions

1. Scalar Fields

$$V(r, \theta, \phi) = \text{scalar field.}$$

Rotate field through angle α about z-axis

$$V'(r, \theta, \phi) = V(r, \theta, \phi - \alpha)$$

$$= \left(1 - \alpha \frac{\partial}{\partial \phi}\right) V = (1 - i \alpha L_z) V$$

- A multipole expansion expresses V as a sum of ~~irreducible~~ quantities which transform like irreducible tensors.

$$V(r, \theta, \phi) = \sum_{lm} V_{lm}(r) \, Y_l^{m*}(\theta, \phi)$$

$$V_{lm} = \int Y_l^{m}(\theta, \phi) \, V(r, \theta, \phi) \, d\Omega .$$

$$e^{i\vec{k}\cdot\vec{r}} = D(\alpha,\beta,\gamma)\, e^{ikz}$$

$$= \sum_{l,m'} i^l (2l+1)\, j_l$$

$$c^*_{l,0}(\theta,\phi) = \sum_{m'} \mathcal{D}^l_{0,m'}(-\gamma,-\beta,-\alpha)\, c^*_{lm'}(\theta,\phi)$$

$$= \sum_{m'} \mathcal{D}^l_{m',0}(\alpha,\beta,\gamma)\, c_{lm'}(\theta,\phi)$$

$$= \sum_{m'} c^*_{lm'}(\beta,\alpha)\, c_{lm'}(\theta,\phi)$$

$$\boxed{\begin{aligned} T'_{kq} &= \sum_p T_{kp}\, \mathcal{D}^k_{pq}(\alpha\beta\gamma) \\ \text{i.e. } c_{l0}(\theta,\phi) &= \sum_p c_{l,p}(\theta,\phi)\, \mathcal{D}^{l*}_{p0}(\alpha\beta\gamma) \\ &= \sum c_{l,p}(\theta,\phi)\, c^*_{l,p}(\beta\alpha) \end{aligned}}$$

Operations with D matrices.

$$\langle I M' | D(\alpha\beta\gamma) | I M \rangle = \mathcal{D}^{I}_{M'M}(\alpha\beta\gamma)$$

$D^{\dagger}$ = adjoint of D.

$$\langle I M' | D^{\dagger}(\alpha\beta\gamma) | I M \rangle = \langle I M | D(\alpha\beta\gamma) | I M' \rangle$$

$$= \left(\mathcal{D}^{I}_{MM'}\right)^{*}$$

Since D is a unitary operator,

$$D^{\dagger}(\alpha\beta\gamma) = D^{-1}(\alpha\beta\gamma) = D(-\gamma\, -\beta\, -\alpha)$$

$$\therefore \left(\mathcal{D}_{MM'}(\alpha\beta\gamma)\right)^{*} = \mathcal{D}_{M'M}(-\gamma\, -\beta\, -\alpha)$$

Example.

$$V = \sum_{lm} \langle V | lm \rangle \langle lm |$$

$$D(\alpha\beta\gamma)\, V = \sum_{\substack{lm \\ m'}} \langle V | lm \rangle \langle lm | \begin{Bmatrix} D(\alpha\beta\gamma) \\ D^{\dagger}(-\gamma-\beta-\alpha) \end{Bmatrix} | l' m' \rangle \langle l' m' |$$

$$= \sum_{\substack{lm \\ m'}} \langle V | lm \rangle \begin{Bmatrix} \mathcal{D}^{l}_{mm'}(\alpha\beta\gamma) \\ \left(\mathcal{D}_{m'm}(-\gamma-\beta-\alpha)\right)^{*} \end{Bmatrix} \langle l' m' |$$

$$D\, | I M \rangle = \sum_{M'} \overset{|I M'\rangle}{\cancel{|I M \rangle}}\, \mathcal{D}^{I}_{M'M*}(\alpha\beta\gamma)$$

$$\langle I M | D^{\dagger}(\alpha\beta\gamma) = \sum_{M'} \left(\mathcal{D}^{I}_{M'M*}(\alpha\beta\gamma)\right)^{*} \langle I M' |$$

$$= \langle I M | D(-\gamma-\beta-\alpha) = \sum_{M'} \mathcal{D}^{I}_{MM'}(-\gamma-\beta-\alpha) \langle I M' |$$

i.e. $V = \sum_{lm} \langle V | lm \rangle \langle lm |$

$$V' = D(\alpha,\beta,\gamma)\, V = \sum_{l^*m'} \sum_{lm} \langle V | lm \rangle \langle lm | D | l^* m' \rangle \langle l^* m' |$$

Put $\langle V' | l^* m' \rangle = \sum_{lm} \langle V | lm \rangle \mathcal{D}^{l}_{m,m'}(\alpha\beta\gamma)$

$$V' = \sum_{l^*m'} \langle V' | l^* m' \rangle \langle l^* m' |$$

Thus $\langle V | lm \rangle$ transforms like an irreducible tensor of rank l under rotations.

$$\text{e.g. } e^{ikz} = \sum_l i^l (2l+1) \cancel{P_l}\, j_l(kr)\, P_l(\cos\theta) \quad \nearrow C^*_{l0}(\theta,\phi)$$

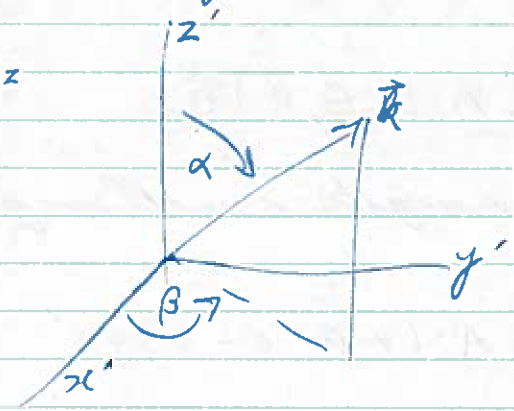
Plane wave travelling in z-direction.

If wave is travelling in arbitrary direction $(\beta,\alpha) = \underset{\sim}{k}$, then

$$e^{i \underset{\sim}{k}\cdot\underset{\sim}{r}} = \sum_{lm'} i^l (2l+1)\, j_l(kr)\, \mathcal{D}^{l}_{0,m'}(-\gamma,-\beta,-\alpha)\, C^*_{lm'}(\theta,\phi)$$

$$= \sum_{lm'} i^l (2l+1)\, j_l(kr)\, C_{lm'}(\beta,\alpha)\, C^*_{lm'}(\theta,\phi)$$

$= D(-\gamma,-\beta,-\alpha)\, e^{ikz}$
$= D^{-1}(\alpha,\beta,\gamma)\, e^{ikz}$



Note that

~~$D|l0\rangle = \sum_{m'} |lm'\rangle\langle lm'|D|l0\rangle$~~

~~$= \sum_{m'} |lm'\rangle \mathcal{D}^{l}_{m',0}$~~

$\langle l0 | D = \sum_{m'} \langle l0 | D | lm' \rangle \langle lm' |$

$= \sum_{m'} \mathcal{D}^{l}_{0,m'} \langle lm' |$

Interelectronic repulsion term

$$\frac{e^2}{|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|} = \sum_l \frac{r_<^{\,l}}{r_>^{\,l+1}} P_l(\cos\theta_{12})$$

$$= \sum_{lm} \frac{r_<^{\,l}}{r_>^{\,l+1}} C_{lm}(\theta_1\phi_1)\, C^*_{lm}(\theta_2\phi_2)$$

Multipole moments.

$$V(r) = \int \frac{\rho(\underset{\sim}{r}')}{|\underset{\sim}{r}-\underset{\sim}{r}'|}\, d\underset{\sim}{r}' = \sum_{lm} \frac{Q_{lm}^*}{r^{l+1}}\, C_{lm}(\theta,\phi)$$

$$Q_{lm} = \int \rho(\underset{\sim}{r}')\, r'^{\,l}\, C_{lm}(\theta',\phi')\, d\underset{\sim}{r}'$$

Q_{lm} = multipole moment of charge distribution.

In q.m. $\rho(r') = \psi^*(r')\psi(r')$

$$Q_{lm} = \langle \psi | r^l C_{lm}(\theta,\phi) | \psi \rangle$$

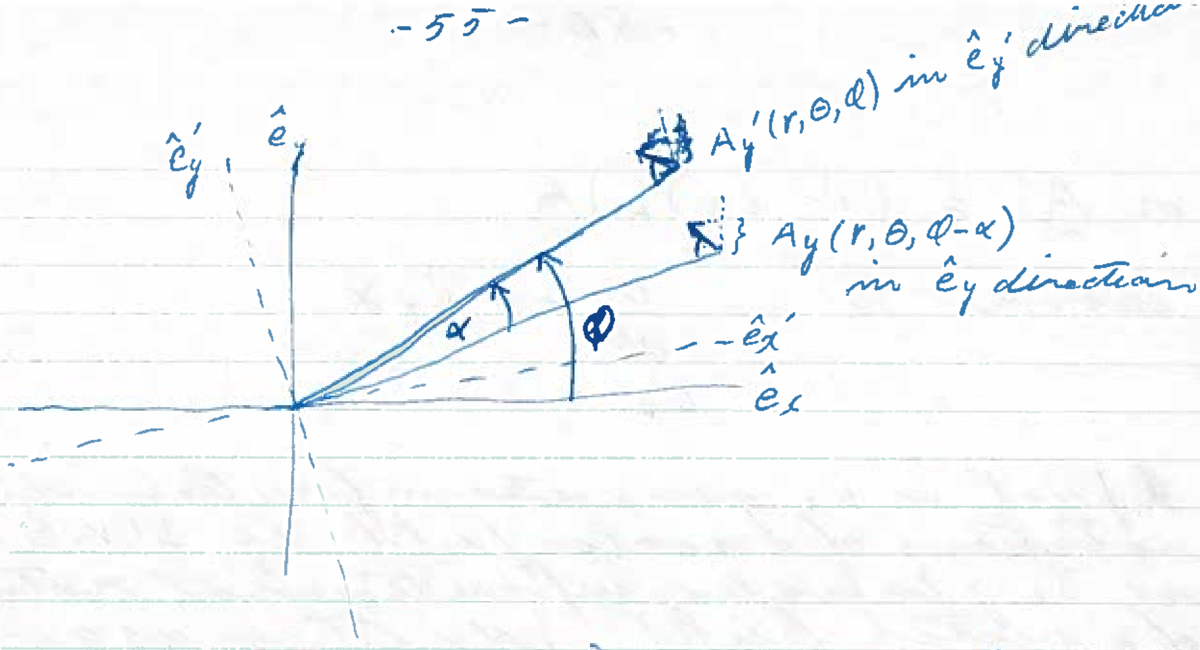
Vector Fields

$$\underset{\sim}{A}(r,\theta,\phi) = \sum_i A_i(r,\theta,\phi)\,\hat{e}_i\,.$$ $\hat{e}_i$ = unit vector.

Rotate field through small angle α about z-axis.

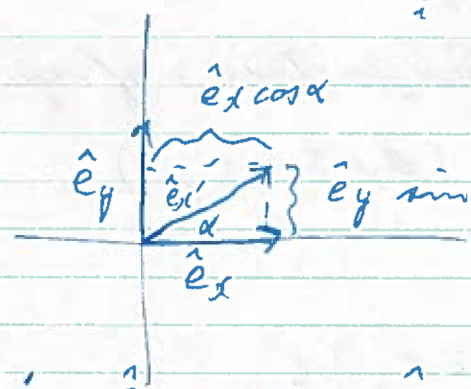
$$\underset{\sim}{A}'(r,\theta,\phi) = \sum_i A_i(r,\theta,\phi-\alpha)\,\hat{e}_i'$$

$\hat{e}_i'$ = unit vector for co-ord. system rotated with the field.



As a result of rotation through angle α, the component $A_y(r,\theta,\phi-\alpha)$ in $\hat{e}_y$ direction is rotated into $A_y'(r,\theta,\phi)$ in $\hat{e}_y'$ direction.

$$\text{ie } \underset{\sim}{A}'(r,\theta,\phi) = \sum_i A_i'(r,\theta,\phi)\,\hat{e}_y'$$
$$= \sum_i A_i(r,\theta,\phi-\alpha)\,\hat{e}_y'$$



$$\hat{e}_x' = \hat{e}_x \cos\alpha + \hat{e}_y \sin\alpha \longrightarrow \hat{e}_x + \alpha\, e_y$$

$$\hat{e}_y' = -\hat{e}_x \sin\alpha + \hat{e}_y \cos\alpha \longrightarrow \hat{e}_y - \alpha\, e_x$$

$$\text{or } \hat{e}_i' = \hat{e}_i + \alpha\, \hat{e}_z \times \hat{e}_i$$

$$\text{Also } A_i(r,\theta,\phi-\alpha) = \left(1 - \alpha\frac{\partial}{\partial\phi}\right) A_i(r,\theta,\phi)$$

$$\therefore \underset{\sim}{A}_i'(r,\theta,\phi) = \underset{\sim}{A}(\underset{\sim}{r},\theta,\phi) + \alpha\left(\hat{e}_z \times \underset{\sim}{A} - \frac{\partial \underset{\sim}{A}}{\partial\phi}\right)$$

or $\underset{\sim}{A}' = (1 - i\alpha J_z)\underset{\sim}{A}$

where $J_z = -i\frac{\partial}{\partial\varphi} + i\,\hat{e}_z \times$

$= L_z + S_z$

just as L_z generates infinitesimal rotations of a scalar field, eg $\psi(r,\theta,\varphi)$, so J_z generates infinitesimal rotations of a vector field. Written as a matrix,

$$1 - i\alpha S_z = \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ operates on } \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

infinitesimal rotation matrix associated with rep. $\mathcal{D}^1$ of rotation group.

$\therefore$ Vector field describes particles of spin 1.

Defining $\hat{e}_\pm = \mp\frac{1}{\sqrt{2}}(\hat{e}_x \pm i\,\hat{e}_y)$

$\hat{e}_0 = \hat{e}_z$

$S_\pm = (S_x \pm i S_y)$ $\qquad S_0 = S_z$

$= i(\hat{e}_x \pm i\,\hat{e}_y)\times$ $\qquad = i\,\hat{e}_z \times$

The $S_\pm$, S_z obey angular momentum commutation relations.

eg. S_+ = raising operator

$S_+\,\hat{e}_0 = \sqrt{2}\,\hat{e}_1$

Special case of $J_+|j\,m\rangle = \{(j+m+1)(j-m)\}^{1/2}|j\,m+1\rangle$

for $j=1$, $m=0$.

$$\underset{\sim}{A} = A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z$$

$$= -A_- \hat{e}_+ - A_+ \hat{e}_- + A_0 \hat{e}_0$$

$$A_\pm = \mp \frac{1}{\sqrt{2}} (A_x \pm i A_y)$$

Each $A_p(r, \theta, \phi)$, $p = \pm 1, 0$ is a scalar field which can be expanded

$$A_p(r, \theta, \phi) = \sum_{lm} \langle A_p | lm \rangle Y_l^{m*}(\theta, \phi)$$

The multipole expansion of $\underset{\sim}{A}$ thus contains products of the form $Y_l^m \hat{e}_n$, corresponding to a reducible representation $\mathcal{D}^l \times \mathcal{D}^1$

The irreducible components are called vector spherical harmonics.

$$\underset{\sim}{Y}_{Ll1}^{M} = \sum_{mn} \langle LM | l1mn \rangle Y_l^m \hat{e}_n .$$

The $\underset{\sim}{Y}_{Ll1}^{M}$ form a complete set for expanding an arbitrary vector field eg photon vector potential.

$$J_z \underset{\sim}{Y}_{Ll1}^{M} = M \underset{\sim}{Y}_{Ll1}^{M}$$

$$J^2 \underset{\sim}{Y}_{Ll1}^{M} = L(L+1) \underset{\sim}{Y}_{Ll1}^{M}$$

$$\int Y_{Ll1}^{M\,*} Y_{L'l'1}^{M'} \, d\Omega = \delta_{ll'} \delta_{LL'} \delta_{MM'} .$$

The $2L+1$ different M components transform amongst themselves under rotations.

e.g. $\underset{\sim}{r} = x e_x + y e_y + z e_z$.

$$= -(r_+^* \hat{e}_-^* + r_-^* \hat{e}_+^*) + r_0 \hat{e}_0 .$$

$$r_q = r\sqrt{\frac{4\pi}{3}}\, Y_1^q$$

$$\therefore \underset{\sim}{r} = r\sqrt{\frac{4\pi}{3}} \sum_m (-1)^m Y_1^{-m} \hat{e}_{+m}$$

$$= -r\sqrt{4\pi} \sum_{mn} \langle 00|1 1 m n\rangle Y_\ell^m \hat{e}_n$$

$$= -r\sqrt{4\pi}\, \underset{\sim}{Y}_{011}^{0}(\theta, \phi)$$

$\therefore J_z \dot{r} = 0$ since the vector field $\underset{\sim}{r}$ is rotationally invariant. It always points radially outward no matter what the orientation

$\underset{\sim}{r}$ is an "isotropic" ~~scalar~~ vector field.

- If Φ_{LM} $M = -L, \dots, L$ is a set of scalar fields forming a spherical tensor of rank L, then the set of vector fields $\underset{\sim}{r}\,\Phi_{LM}$ also transforms as a spherical tensor of rank L.

Specifically,

$$\frac{\underset{\sim}{r}}{r} Y_L^M = -\left[\frac{L+1}{2L+1}\right]^{1/2} \underset{\sim}{Y}_{L\,L+1\,1}^M + \left[\frac{L}{2L+1}\right]^{1/2} \underset{\sim}{Y}_{L\,L-1\,1}^M .$$

Roughly speaking, $\vec{r}$ is a "scalar" vector field.

e.g. under rotations $D(\underset{\sim}{r}\, Y_L^M) = \sum_{M'} \mathcal{D}^L_{MM'} (\underset{\sim}{r}\, Y_L^{M'})$

The vector components are $\underset{\sim}{Y}_{L\ell 1}^M = \sum_{mn} \langle LM|\ell 1 m n\rangle Y_\ell^m \hat{e}_n$