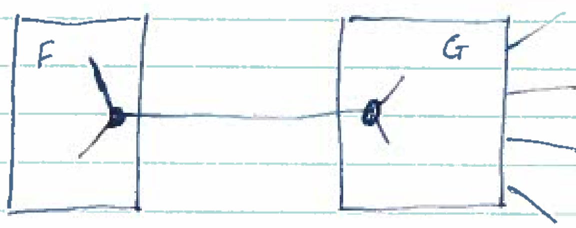
Rules for Transforming Graphs

Reducible Graphs

- Suppose a graph can be split into two parts F and G connected by one, two or three lines, such that
1. F has no external lines, other than the connecting lines.
2. Every internal line of F contains exactly one arrow. (Arrows on connecting lines are included in G.)
- Such graphs reduce to simple products of parts according to the following rules

I.



(G can be nothing at all).

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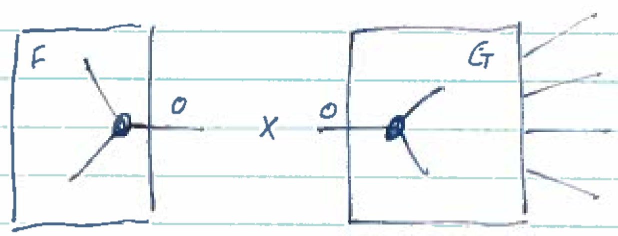
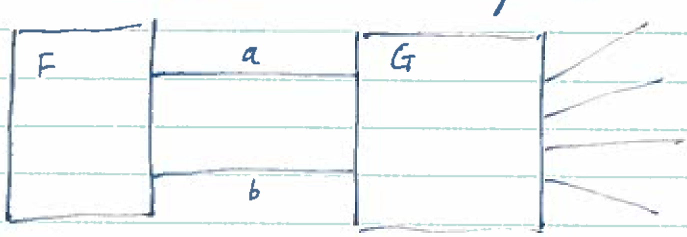
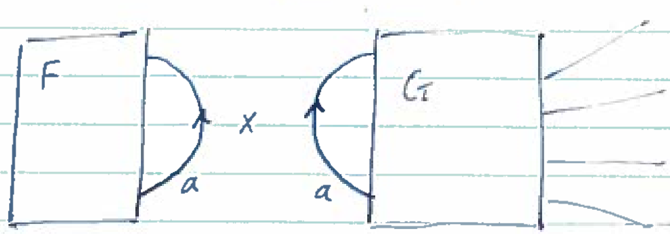


e.g. If a graph has only a single external line, that line must have $j = 0$.

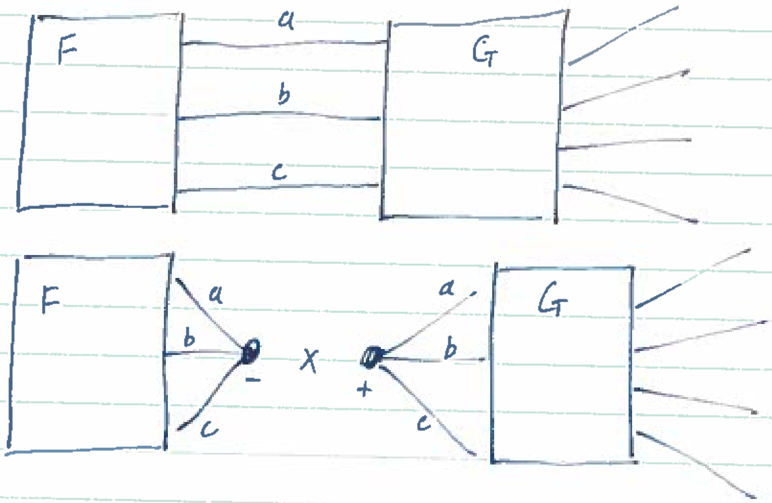
II.



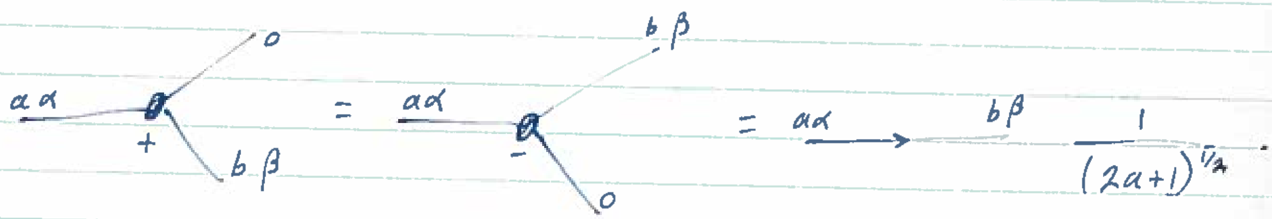
$= \dfrac{\delta_{a,b}}{2a+1}$



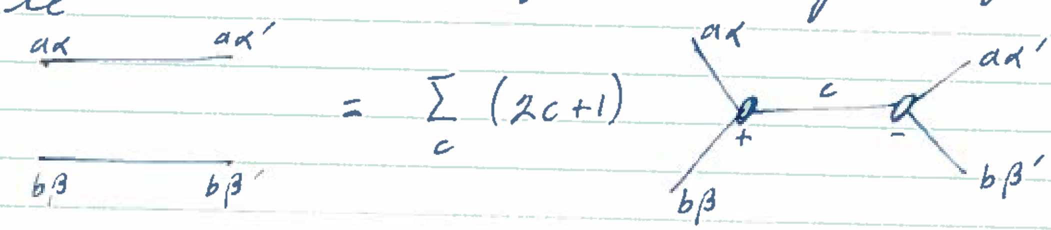
III



- Note that II and I are special cases of III if we use the rule



- Graphs connected by four or more lines can be separated into sums of products by use of the rule

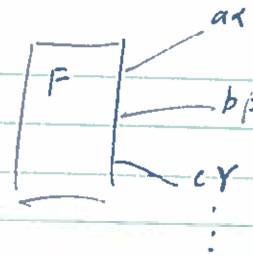


i.e. $\delta_{\alpha,\alpha'}\,\delta_{\beta,\beta'} = \sum_c (2c+1)\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} a & b & c \\ \alpha' & \beta' & \gamma \end{pmatrix}$

e.g. repeated use of this rule reduces III and II to I.

- Exercise, using the above rule, reduce III and II to I.

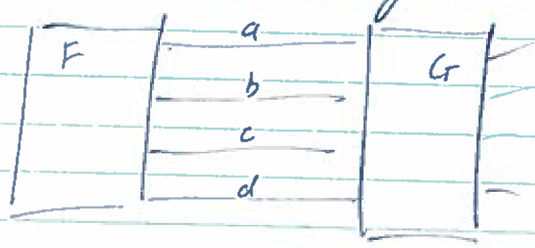
I is true because normal graphs always result from the repeated coupling of angular momenta to form a scalar.

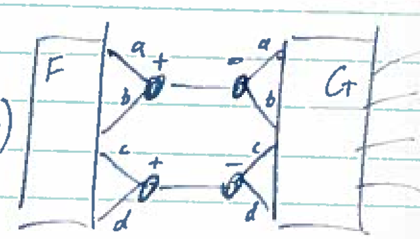
Define $F\begin{pmatrix} a & b & c & \dots \\ \alpha & \beta & \gamma & \end{pmatrix} =$ 

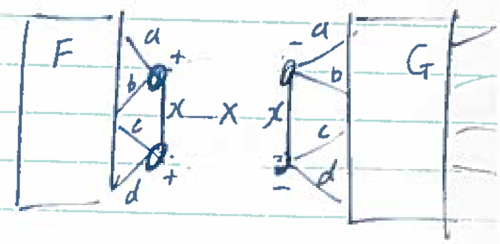
Then scalar $= \sum_{\alpha,\beta,\gamma} F\begin{pmatrix} a & b & c & \dots \\ \alpha & \beta & \gamma & \end{pmatrix} \phi_1(a,\alpha)\,\phi_2(b,\beta)\,\phi_3(c,\gamma)\dots$

A single angular momentum function $\phi_1(a\alpha)$ can be coupled to form a scalar only if $a = 0$.

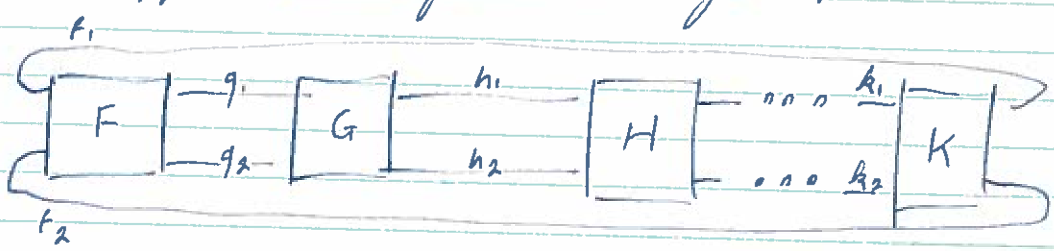
<u>Four Connecting Lines</u>



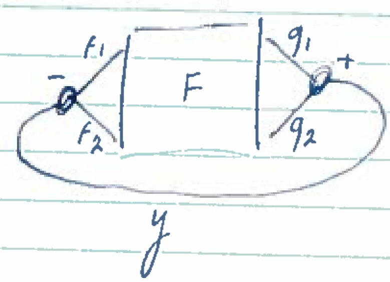
$= \sum_{x,y} (2x+1)(2y+1)$ 

$= \sum_{x} (2x+1)$ 

Repeated application of this rule gives



$= \sum_y (2y+1)\, F(y)\, G(y)\, H(y) \dots K(y)$

where $F(y) =$  etc.

Coupling of Three Angular Momenta.

$$|j_1 m_1, j_2 m_2, j_3 m_3\rangle = |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle$$

uncoupled rep.

Find linear combinations which diagonalize

$$\underset{\sim}{J}^2 = (\underset{\sim}{j}_1 + \underset{\sim}{j}_2 + \underset{\sim}{j}_3)^2, \quad J_z = j_{1,z} + j_{2,z} + j_{3,z}$$

Unlike previous case, the same J may be formed in more than one way.

$$|j_1 j_2 J_{12} M_{12}\rangle = \sum_{m_1 m_2} |j_1 m_1\rangle |j_2 m_2\rangle \langle j_1 j_2 m_1 m_2 | J_{12} M_{12}\rangle$$

$$\begin{array}{lcl} J_{12} & & \\ j_1 + j_2 & & \left\{\begin{array}{l} j_1 + j_2 + j_3 \\ j_1 + j_2 + j_3 - 1 \\ \vdots \\ |j_1 + j_2 - j_3| \end{array}\right. \\ j_1 + j_2 - 1 & + j_3 & \\ \vdots & & \\ |j_1 - j_2| & & \end{array}$$

eg $j_1 = 1, j_2 = 1, j_3 = 1$

$$\begin{array}{cccccl} \underset{\sim}{j}_1 + \underset{\sim}{j}_2 & = & \underset{\sim}{J}_{12} & + & \underset{\sim}{j}_3 & = \quad J \\ p + p & & 2 & & D & \left\{\begin{array}{l} 3 \\ 2 \\ 1 \end{array}\right. \\ & & 1 & & P & \left\{\begin{array}{l} 2 \\ 1 \\ 0 \end{array}\right. \\ & & 0 & & S & \left\{\begin{array}{l} 1 \\ \end{array}\right. \end{array}$$

1 - F

2 - D's

3 - P's

1 - S's

couple with j_3

$$|(j_1 j_2) J_{12}, j_3; J M\rangle = \sum_{M_{12} m_3} |\{j_1 j_2\} J_{12} M_{12}\rangle |j_3 m_3\rangle \langle J_{12} j_3 M_{12} m_3 | J M\rangle$$

This state is an eigenfcn of J_{12}^2 (but not $J_{12,z}$). Value of J_{12} distinguishes states with same $J\ M$.

.. alternatively, couple $\underset{\sim}{j_2} + \underset{\sim}{j_3} = \underset{\sim}{J}_{23}$ and couple resultant to $\underset{\sim}{j_1}$.

e.g. $j_1 = 1$, $j_2 = 2$, $j_3 = 3$.

	J_{12}		J	$J =$
	3		6, 5, 4, 3, 2, 1, 0	1 - 6
				2 - 5
				3 - 4
				3 - 3
		+ 3 =		3 - 2
1 + 2 =	2		5, 4, 3, 2, 1, ~~0~~	2 - 1
				1 - 0
	1		4, 3, 2, ~~1~~	

	J_{23}		J
	5		$\begin{cases}6\\5\\4\end{cases}$
	4		$\begin{cases}5\\4\\3\end{cases}$
$2+3$ =	3	$+1$ =	$\begin{cases}4\\3\\2\end{cases}$
	2		$\begin{cases}3\\2\\1\end{cases}$
	1		$\begin{cases}2\\1\\0\end{cases}$

J =
1 - 6
2 - 5
3 - 4
3 - 3
3 - 2
2 - 1
1 - 0

In above coupling scheme.

$$|j_2 j_3 J_{23} M_{23}\rangle = \sum_{m_2 m_3} |j_2 m_2\rangle |j_3 m_3\rangle \times \langle j_2 j_3 m_2 m_3 | J_{23} M_{23}\rangle$$

$$|j_1 (j_2 j_3) J_{23}; JM\rangle = \sum_{m_1 M_{23}} |j_1 m_1\rangle |j_2 j_3 J_{23} M_{23}\rangle \times \langle j_1 J_{23} m_1 M_{23} | JM\rangle$$

The two coupling schemes represent the same set of states and must be related ~~to~~ by a unitary transform (similarity transform).

$$|(j_1 j_2) J_{12}, j_3; JM\rangle = \sum_{J_{23}} |j_1 (j_2 j_3) J_{23}; JM\rangle \times \langle j_1, (j_2 j_3) J_{23}; J | (j_1 j_2) J_{12}, j_3; J\rangle$$

The transform. coefs. are independent of M.

General form

$$|\alpha J M\rangle = \sum_\beta |\beta J M\rangle \langle \beta J M | \alpha J M\rangle$$

apply $J_+ = J_x + i J_y$

$$|\alpha J \, M+1\rangle = \sum_\beta |\beta J \, M+1\rangle \langle \beta J M | \alpha J M\rangle$$

$\therefore \langle \beta J M | \alpha J M\rangle = \langle \beta J | \alpha J\rangle$ is rotationally invariant (i.e. a scalar)

The coef. of $|j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle$ on LHS is

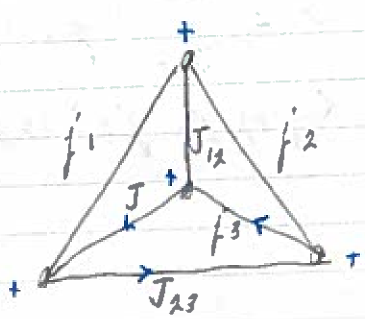
$$\langle j_1 j_2 m_1 m_2 | J_{12} M_{12}\rangle \langle J_{12} j_3 M_{12} m_3 | J M\rangle$$

on RHS is

$$\sum_{J_{23}} \langle j_2 j_3 m_2 m_3 | J_{23} M_{23}\rangle \langle j_1 J_{23} m_1 M_{23} | J M\rangle$$
$$\times \langle j_1, (j_2 j_3) J_{23}; J | (j_1 j_2) J_{12}, j_3 ; J\rangle$$

$$\langle (j_1 j_2) J_{12}, j_3 ; J | j_1 (j_2 j_3) J_{23} ; J\rangle$$

$$= (-1)^{j_1+j_2+j_3+J} \left[(2J_{12}+1)(2J_{23}+1) \right]^{1/2} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & J & J_{23} \end{Bmatrix}$$



Four angular Momenta:

e.g. $LS \to jj$ coupling transform.

Couple I

$$\left.\begin{array}{l} l_1 + l_2 \longrightarrow L \\ s_1 + s_2 \longrightarrow S \end{array}\right\} \quad L + S \longrightarrow J$$

II

$$\left.\begin{array}{l} l_1 + s_1 \longrightarrow j_1 \\ l_2 + s_2 \longrightarrow j_2 \end{array}\right\} \quad j_1 + j_2 \longrightarrow J$$

I $\quad |(l_1 l_2)L, (s_1 s_2)S; (LS)J\rangle$

$$= \sum |l_1 m_1\rangle |l_2 m_2\rangle |s_1 \mu_1\rangle |s_2 \mu_2\rangle$$

$$\langle l_1 l_2 m_1 m_2 | L M_L\rangle \langle s_1 s_2 \mu_1 \mu_2 | S M_S\rangle$$

$$\langle L S M_L M_S | J M\rangle$$

II $\quad |(l_1 s_1) j_1, (l_2 s_2) j_2; (j_1 j_2) J\rangle$

$$= \sum |l_1 m_1\rangle |l_2 m_2\rangle |s_1 \mu_1\rangle |s_2 \mu_2\rangle$$

$$\langle l_1 s_1 m_1 \mu_1 | j_1 M_{j_1}\rangle \langle l_2 s_2 m_2 \mu_2 | j_2 M_{j_2}\rangle$$

$$\langle j_1 j_2 M_{j_1} M_{j_2} | J M\rangle$$

I and II are connected by a unitary transform.

$$|(l_1 s_1) j_1, (l_2 s_2) j_2; (j_1 j_2) JM\rangle$$

$$= \sum_{LS} |(l_1 l_2)L, (s_1 s_2)S; (LS)JM\rangle$$

$$\langle (l_1 l_2)L, (s_1 s_2)S; (LS)J | (l_1 s_1) j_1, (l_2 s_2) j_2; (j_1 j_2) J\rangle$$

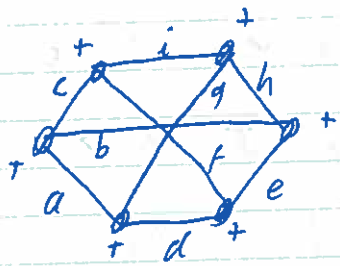
as before

$$\langle (l_1 l_2)L, (s_1 s_2)S; (\cancel{j_1}\,\cancel{j_2})^{L\ S} J \,|\, (l_1 s_1)j_1, (l_2 s_2)j_2; (j_1 j_2)J \rangle$$

$$= [(2L+1)(2S+1)(2j_1+1)(2j_2+1)]^{1/2} \begin{Bmatrix} l_1 & l_2 & L \\ s_1 & s_2 & S \\ j_1 & j_2 & J \end{Bmatrix}$$

9-j symbol is invariant under ~~any permutation~~ or transposition.

$$\begin{Bmatrix} a & b & e \\ c & d & e \\ f & f & 0 \end{Bmatrix} = \frac{(-1)^{b+c+e+f}}{[(2e+1)(2f+1)]^{1/2}} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$$

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} =$$ 

Permutations multiply by $(-1)^{\Sigma j}$