The Dirac Equation

- Instead of squaring the energy expression, we can attempt to linearize the square root. To do this, define operators $\alpha_1, \alpha_2, \alpha_3$ and β such that

$$\left[\sum_{i=1}^{3} p_i^2 + m^2c^2\right]^{1/2} = \sum_{i=1}^{3} \alpha_i p_i + \beta mc$$

The α_i and β are noncommutative operators determined by

$$p_1^2 + p_2^2 + p_3^2 + m^2c^2 = \left[\sum_{i=1}^{3} \alpha_i p_i + \beta mc\right]\left[\sum_{j=1}^{3} \alpha_j p_j + \beta mc\right]$$

for any p_i and m. The RHS gives

$$\begin{aligned} RHS = {} & \alpha_1^2 p_1^2 + \alpha_2^2 p_2^2 + \alpha_3^2 p_3^2 + \beta^2 m^2 c^2 \\ & + (\alpha_1\alpha_2 + \alpha_2\alpha_1) p_1 p_2 + \cdots \\ & + (\alpha_1\beta + \beta\alpha_1) p_1 mc + \cdots \end{aligned}$$

Comparing with the LHS gives

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1$$

$$\alpha_i\alpha_k + \alpha_k\alpha_i = 0 \qquad i, k = 1, 2, 3 \quad i \neq k$$

$$\alpha_i\beta + \beta\alpha_i = 0 \qquad i = 1, 2, 3$$

Let us assume for the moment that such operators exist. We then obtain the linearized energy expression

$$E - c\sum_{i=1}^{3} \alpha_i p_i - \beta mc^2 = 0$$

Or, with the usual operator replacements

$$\left\{ i\hbar \frac{\partial}{\partial t} - \frac{c\hbar}{i} \sum_{i=1}^{3} \alpha_i \frac{\partial}{\partial x_i} - \beta mc^2 \right\} \Psi = 0$$

Define $H = \frac{c\hbar}{i} \sum_{i=1}^{3} \alpha_i \frac{\partial}{\partial x_i} + \beta mc^2$

Then $i\hbar \frac{\partial \Psi}{\partial t} = H \Psi$. Dirac equ'n.

This is now of first order in both space and time variables.

Symmetrized Form

- Multiply by $\frac{1}{\hbar c} \beta$ and define

$$\gamma_k = -i\beta\alpha_k, \quad \gamma_4 = \beta, \quad \kappa = m_0 c/\hbar$$

The Dirac equ'n. then becomes

$$(\gamma_\mu \partial_\mu + \kappa) \Psi = 0. \qquad (\text{sum } \mu = 1, \dots, 4)$$

with $\partial_k = \frac{\partial}{\partial x_k}$, $\partial_4 = \frac{\partial}{\partial x_4} = \frac{1}{ic} \frac{\partial}{\partial t}$

The γ operators satisfy

$$\left. \begin{array}{l} \gamma_\mu^2 = 1 \\ \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 0 \end{array} \right\} \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu,\nu}.$$

- The simplest objects which obey these commutation relations are 4×4 matrices

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{e.g.}\quad \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Free Particle Solutions

- Since the Dirac equation is a 4×4 matrix equation, ψ must be a 4-component column vector

$$\psi(\vec{r},t) = \begin{pmatrix} \psi_1(\vec{r},t) \\ \psi_2 \quad " \\ \psi_3 \quad " \\ \psi_4 \quad " \end{pmatrix}$$

The Dirac equation then becomes a set of 4 coupled 1st-order equations

- Look for plane wave solutions of the form

$$\psi_j(\vec{r},t) = u_j e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad j = 1, \ldots, 4.$$

with $\vec{p} = \hbar\vec{k}$ and $E = \hbar\omega$. Substituting into

$$(E - c\vec{\alpha}\cdot\vec{p} - \beta mc^2)\psi = 0$$

yields the algebraic equations

$$(E - mc^2)u_1 - cp_z u_3 - c(p_x - ip_y)u_4 = 0$$

$$(E - mc^2)u_2 - c(p_x + ip_y)u_3 + cp_z u_4 = 0$$

$$(E + mc^2)u_3 - cp_z u_1 - c(p_x - ip_y)u_2 = 0$$

$$(E + mc^2)u_4 - c(p_x + ip_y)u_1 + cp_z u_2 = 0$$

These are homogeneous equations for the u_j. The determinant is

$$(E^2 - m^2c^4 - c^2p^2)^2 = 0$$

in agreement with the relativistic connection between energy and momentum.

- we expect in all 4 solutions.

$$E_+ = (c^2p^2 + m^2c^4)^{1/2} \quad \text{Two sol'ns.}$$
$$E_- = -(c^2p^2 + m^2c^4)^{1/2} \quad \text{" \quad "}$$

choosing $E = E_+$ gives

$$\begin{array}{ccl} 1 & 0 & \text{large} \\ 0 & 1 & \text{components} \\ \dfrac{cp_z}{E_+ + mc^2} & \dfrac{c(p_x - ip_y)}{E_+ + mc^2} & \text{small} \\ \dfrac{c(p_x + ip_y)}{E_+ + mc^2} & -\dfrac{cp_z}{E_+ + mc^2} & \text{components} \end{array}$$

Choosing $E = E_-$ gives

$$
\begin{array}{ccl}
\dfrac{cp_z}{E_- - mc^2} & \dfrac{c(p_x + i p_y)}{E_- - mc^2} & \text{small} \\
\dfrac{c(p_x - i p_y)}{E_- - mc^2} & -\dfrac{cp_z}{E_- - mc^2} & \text{components.} \\
1 & 0 & \text{large} \\
0 & 1 & \text{components}
\end{array}
$$

To normalize so that $\psi^\dagger \psi = 1$, multiply by

$$
\frac{1}{\left[1 + \dfrac{c^2 p^2}{(E_+ + mc^2)^2}\right]^{1/2}}
$$

<u>Charge and Current Densities</u>

- The Hermitian adjoint eqn. is

$$
-i\hbar \frac{\partial \psi^\dagger}{\partial t} - i\hbar c (\nabla \psi^\dagger) \cdot \vec{\alpha} + \psi^\dagger \beta mc^2 = 0
$$

Multiply on the right and subtract

$$
\psi^\dagger \left(i\hbar \frac{\partial}{\partial t} + i\hbar c \alpha \cdot \nabla - \beta mc^2 \right) \psi = 0
$$

to obtain

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0
$$

with $\rho = \psi^\dagger \psi$, $\vec{j} = c\psi^\dagger \vec{\alpha} \psi$

In the N.R. limit
$$\vec{j} \to \frac{\hbar}{2mi}\left[\psi^*\nabla\psi - (\nabla\psi^*)\psi\right] \quad \text{as expected.}$$

$\vec{\alpha}$ itself can be interpreted as the particle velocity operator. This follows from e.g.

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x, H] = c\alpha_x .$$

<u>Interaction with External Fields</u>

- As usual, put $c\vec{p} \to c\vec{p} - e\vec{A}$, $E \to E - e\phi$ to obtain

$$[E - e\phi - \vec{\alpha}\cdot(c\vec{p} - e\vec{A}) - \beta mc^2]\psi = 0.$$

This equation is difficult to interpret as it stands because of the way it mixes large and small components.
- To obtain an equation which acts primarily on large components, multiply by
$$[E - e\phi + \vec{\alpha}\cdot(c\vec{p} - e\vec{A}) + \beta mc^2]$$
to obtain

$$\{(E-e\phi)^2 - [\vec{\alpha}\cdot(c\vec{p}-e\vec{A})]^2 - m^2c^4$$
$$- [E - e\phi, \vec{\alpha}\cdot(c\vec{p} - e\vec{A})]\}\psi = 0$$

Note that $\vec{\alpha}\beta + \beta\vec{\alpha} = 0$.

- Use the identity

$$(\vec{\alpha}\cdot\vec{B})(\vec{\alpha}\cdot\vec{C}) = \vec{B}\cdot\vec{C} + i\vec{\Sigma}\cdot\vec{B}\times\vec{C}$$

where $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

to obtain $[\vec{\alpha}\cdot(c\vec{p}-e\vec{A})]^2 = (c\vec{p}-e\vec{A})^2 + i\vec{\Sigma}\cdot(c\vec{p}-e\vec{A})\times(c\vec{p}-e\vec{A})$

Also $(c\vec{p}-e\vec{A})\times(c\vec{p}-e\vec{A}) = -ce(\vec{A}\times\vec{p}+\vec{p}\times\vec{A})$

$$= ie\hbar c\,\nabla\times\vec{A} = ie\hbar c\vec{B}.$$

Thus $[\vec{\alpha}\cdot(c\vec{p}-e\vec{A})]^2 = (c\vec{p}-e\vec{A})^2 - e\hbar c\vec{\Sigma}\cdot\vec{B}$.

The remaining commutator term is

$-[E-e\phi, \vec{\alpha}\cdot(c\vec{p}-e\vec{A})]$ (where $E = i\hbar\frac{\partial}{\partial t}$)

$$= e\vec{\alpha}\cdot[E,\vec{A}] + ce\vec{\alpha}\cdot[\phi,\vec{p}]$$

$$= ie\hbar\vec{\alpha}\cdot\frac{\partial\vec{A}}{\partial t} + ie\hbar c\vec{\alpha}\cdot\nabla\phi = -ie\hbar c\vec{\alpha}\cdot\vec{E}$$

since $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$.

The Dirac eqn. thus becomes

$$[(E-e\phi)^2 - (c\vec{p}-e\vec{A})^2 - m^2c^4 + e\hbar c\vec{\Sigma}\cdot\vec{B}$$

$$- ie\hbar c\vec{\alpha}\cdot E]\psi = 0.$$

Only the last term now couples large and small components. To study the nonrelativistic limit, put

$$E = E' + mc^2 .$$

Then $(E - e\phi)^2 - m^2c^4 \simeq 2mc^2(E' - e\phi)$
and

$$E'\psi = \left\{ \frac{1}{2m}\left(\vec{p} - \frac{e}{c}A\right)^2 + e\phi - \frac{e\hbar}{2mc}\vec{\Sigma}\cdot\vec{B} + \frac{ie\hbar}{2mc}\vec{\alpha}\cdot\vec{E} \right\}\psi$$

Interpretation

- The $-\frac{e\hbar}{2mc}\vec{\Sigma}\cdot\vec{B}$ term clearly represents the interaction of a particle with magnetic moment $\frac{e\hbar}{2mc}\vec{\Sigma}$ with a magnetic field.

- The $\vec{\alpha}\cdot\vec{E}$ term is smaller by a factor of $(v/c)^2$ since $\vec{\alpha}$ mixes large and small components. Recall that $c\vec{\alpha}$ is the velocity operator. For central fields, the $\vec{\alpha}\cdot\vec{E}$ term is related to the spin-orbit interaction, as will soon be shown.

$$2\left(1 + \frac{\alpha}{2\pi}\right) \simeq 2.00232\ldots$$